MATHEMATICS FOR ELECTRICIANS

The quality of the materials used in the manufacture of this book is governed by continued postwar shortages.

Mathematics

FOR ELECTRICIANS

BY

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PREFACE TO THE SECOND EDITION

The principal change made in this edition is the addition of several chapters on alternating-current problems. A considerable variety of alternating-current problems has now been provided.

Several chapters of the original text have been completely rewritten, and minor changes have been made in others. The number of problems taken from, or adaptable to, the field of radio electricity has been increased considerably.

M. H. Kuehn.

PREFACE TO THE FIRST EDITION

This book aims to satisfy the need for a volume in which problems in practical electricity are combined with the principles of mathematics used in their solution. This need is general among vocational-school teachers and has been expressed to the author by many of them.

All groups of practical problems in the text are preceded by a study of the principles of mathematics required in their solution. The greater part of the material included in the text has been used and developed in the class room. Due to the variety of problems included, teachers should have no difficulty in selecting such material as is suited to their requirements, and they will find the material so arranged that many problems may be omitted, if desired, without destroying the continuity of the work.

Aside from its usefulness in the class room, this textbook should be of material aid to the mechanic who is unable to attend school and who wishes to be informed on the mathematics of his trade. Such a student will find, owing to numerous illustrative examples for which the solutions are given, that he will be able to master the mathematics required in the solution of practical electrical problems through a process of self-instruction.

Mathematical principles have been stressed in the sample solutions of all of the applied problems. Special attention is called to the method of finding the resistance of a parallel group, a procedure explained in Sec. 33. This was developed in the class room, where it was found that the students were able to obtain quicker and more accurate results when using this method than they were able to obtain by the use of any other.

The author will be pleased to receive criticisms regarding any part of the book and will appreciate being informed of any errors which may have escaped his notice.

He wishes to acknowledge his indebtedness to Dr. Elmer S. Pierce, principal of Seneca Vocational High School, whose suggestions prompted the beginnings from which this book has grown and through whose cooperation its final publication has been made possible. His thanks are duc also to Mr. A. D. Dimmick, head of the Academic Department, through whose active cooperation the original mimeographed copy, upon which this book is largely based, was secured.

M. H. KUEHN.

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MATHEMATICS FOR ELECTRICIANS

CHAPTER I

SUBSTITUTION

1. Numbers.—In arithmetic we deal usually only with numbers whose value, or size, is known. It is often necessary, however, to represent numbers or quantities whose value we do not know. Whenever we use a number to represent a quantity whose value we do not know, we call such a number an "unknown" number. We deal, therefore, with two kinds of numbers, known and unknown. The branch of mathematics which uses letters or other symbols to represent unknown quantities is known as *algebra*. We shall have much to do with algebra in this text.

2. Known Quantities.—Known quantities are represented by figures. As we shall learn later, a known quantity may be either positive or negative.

3. Unknown Quantities.—Unknown quantities are represented by letters or combinations of figures and letters. They are also known as *literal quantities* and may be either positive or negative.

4. Substitution.—The process of replacing any number by its equal is called "substitution." For example, in this chapter numerical values are assigned to certain letters, and where these letters occur in the problem they are to be replaced by the given numerical values; in other words, the numerical values are to be substituted for the letters.

5. Signs of Operation.—The symbols $+$, $-$, $\times$, and $\div$ are used in algebra as they are in arithmetic. In algebra, however,

the multiplication sign is usually omitted. For example, $3a$ means 3 times a, $6(5)$ means 6 times 5, $3a(4b)$ means $3a$ times $4b$, etc. It is often convenient to use the dot as a sign of multiplication; for example, $3 \cdot 8$ means 3 times 8.

$\therefore$ is a symbol which is often used. It means "therefore."

Example 1. If $a = 5$, $b = 4$, and $c = 9$ find the value of $3ac - b$.

Solution: Substitute the given numerical values for the letters and solve, as follows:

$$\begin{aligned} 3ac - b &= 3(5)9 - 4 \\ &= 135 - 4 \\ &= 131 \textit{ Ans.} \end{aligned}$$

Problems

Find the value of each of the following expressions, using the equivalents $a = 2$, $b = 4$, and $c = 7$:

1. $a + b$

2. $a + b + c$

3. $c - a + b$

4. $c - a - b$

5. $5a - b$

6. $7b - c$

7. $3b + a$

8. $4a - b$

9. $b - a$

10. $7b - 16$

11. $3a(4b)$

12. $5a(3b)$

13. $6c - 38$

14. $5b - a - c$

15. $3abc$

16. $7bc - 98$

17. $abc + 44$

18. $\dfrac{12a}{b}$

19. $\dfrac{5a}{2b}$

20. $\dfrac{3a + 16}{4a}$

21. $\dfrac{9a - 12}{2b}$

22. $\dfrac{6bc}{3a}$

23. $3a\left(\dfrac{6b}{3b}\right)$

24. $\dfrac{2ac - 4}{4a}$

Common fractions can be combined by addition or subtraction only when they have the same denominators. Before combining fractions having different denominators, it is necessary to change them to equivalent fractions having equal denominators.

Example 2. Find the value of $2/3 + 3/4$.

Solution:
$$\begin{aligned} 2/3 + 3/4 &= 2/3 \times 4/4 + 3/4 \times 3/3 \\ &= 8/12 + 9/12 \\ &= \frac{8 + 9}{12} = \frac{17}{12} = 1\frac{5}{12} \textit{ Ans.} \end{aligned}$$

This process should be simplified by elimination of the first two steps, as follows:

$$\frac{2}{3} + \frac{3}{4} = \frac{8+9}{12} = \frac{17}{12} = 1\frac{5}{12} \textit{ Ans.}$$

The fraction to the right of the first equal sign is determined by taking for its denominator the least common multiple of 3 and 4, the denominators of the two fractions which are to be added. The figure 8 is obtained by dividing the denominator of the first fraction into 12 and multiplying the results by the numerator 2. Similarly, by dividing 4 into 12 and multiplying the result by 3, the numerator of the second fraction, we obtain the figure 9.

Example 3. If $a = 1, b = 2, c = 3, d = 4$, find the value of $\frac{a}{b} + \frac{bc}{7} - \frac{c}{d}$.

Solution: Step 1: Substitute the numerical equivalents for the letters. This gives

$$\frac{a}{b} + \frac{bc}{7} - \frac{c}{d} = \frac{1}{2} + \frac{2(3)}{7} - \frac{3}{4}$$

Step 2: Find the least common multiple of the denominators 2, 7, and 4. This is 28.

Step 3: Using 28 as the common denominator, proceed as in Ex. 3. The complete solution then will be as follows:

$$\begin{aligned}\frac{a}{b} + \frac{bc}{7} - \frac{c}{d} &= \frac{1}{2} + \frac{2(3)}{7} - \frac{3}{4}\\ &= \frac{14+24-21}{28}\\ &= 1\tfrac{7}{28} \textit{ Ans.}\end{aligned}$$

Problems

Evaluate each of the following, using the equivalents $a = 3$, $b = 5$, $c = 7$, $d = 8$, $e = 2$:

1. $\frac{1}{a} + \frac{2}{b} + \frac{3}{c}$
2. $\frac{1}{c} + \frac{1}{d} + \frac{1}{e}$
3. $\frac{cde}{4} + \frac{ace}{6}$
4. $\frac{2bc}{a} - \frac{4ab}{c}$
5. $\frac{cd}{a} + \frac{be}{3} - \frac{3bd}{c}$
6. $\frac{4cd}{e} - \frac{4bc}{5} + \frac{1}{5}$
7. $\frac{cd}{4a} - \frac{ce}{3b} - \frac{4c}{e}$
8. $\frac{4b}{a} + \frac{bc}{b} + \frac{7c}{c}$
9. $\frac{3cd}{7} - \frac{4ed}{c} + \frac{1}{7}$
10. $\frac{7e}{4a} - \frac{5}{b} + \frac{1}{3d}$
11. $\frac{c}{4e} - \frac{a}{2d} + \frac{b}{a}$
12. $\frac{3d}{2c} + \frac{3ea}{4} - \frac{4a}{c}$

13. $\frac{c}{a} - \frac{a}{2e} - \frac{b}{2a}$

14. $\frac{2b}{3} - \frac{2e}{c} - \frac{29e}{ac}$

15. $\frac{bc}{ae} + \frac{7a}{d} + \frac{bde}{4a}$

16. $\frac{a}{c} + \frac{3e}{c} + \frac{c}{d} - \frac{3a}{cd}$

Whenever a series of operations involving addition, subtraction, and multiplication or division occurs, the operations must be performed in a definite order.

For example, $6 + 3 \times 4$ means 6 plus the *product* of 3 and 4, that is, $6 + 3 \times 4 = 6 + 12$.

$8 - 12 \div 4$ means 8 minus the *quotient* of 12 divided by 4, that is,

$$8 - 12 \div 4 = 8 - 3$$

Example 4. Find the value of $8 + 7(16) - 5 + 18(7) \div 9$.

Solution: $8 + 7(16) - 5 + 18(7) \div 9$
$= 8 + 112 - 5 + 126 \div 9$
$= 8 + 112 - 5 + 14$
$= 129$ *Ans.*

Example 5. If $a = 1, b = 2, c = 3, d = 4$ find the value of $\frac{ab + c - d}{cd - 1}$.

Solution 1: $\frac{ab + c - d}{cd - 1} = \frac{1(2) + 3 - 4}{3(4) - 1}$
$= \frac{2 + 3 - 4}{12 - 1}$
$= 1/11$ *Ans.*

Note that the numerator and the denominator in this problem must be worked out separately. Note also that the figure 4, which occurs in both numerator and denominator, cannot be cancelled. Beginners should not attempt cancellation so long as plus or minus signs are present in the numerator or in the denominator of a fraction.

Problems

Evaluate each of the following, using the equivalents $a = 3$, $b = 5$, $c = 7$, $d = 8$, $e = 2$:

1. $7 + 4(9) - 3(5)$

2. $27 - 5(4) + 8 - 15 \div 3$

3. $\frac{34 + 12 \div 2}{5(2)} + \frac{8(7) - 6}{7 + 3}$

4. $\frac{363 \div 11}{3} - \frac{24(5) - 36}{12}$

5. $\frac{4abc - 5ad}{ad}$

6. $\frac{ab + d - c}{cd - 8}$

7. $8a + 2ae - \frac{13c - 11d}{3}$

8. $\frac{3b + 8c - d}{a + b + c}$

9. $\dfrac{3be - ac + 2bd - cd}{a + c + e}$

10. $\dfrac{3cd}{4e} - \dfrac{bc - 4e - 1}{13}$

11. $\dfrac{2ad - 3de + 5bc}{ac + ce}$

12. $ace + \dfrac{ce - 2ab \div b}{2e}$

13. $0.7 + 0.4(0.9) - 0.4(0.6)$

14. $12 - 2 \div 0.5 + 3(0.14)$

15. $8 - 0.7(1.6) + 3 \div 0.02$

16. $\dfrac{0.4(3) - 0.5 + 0.6(2)}{5(0.03) + 0.23}$

Frequently parentheses are used to segregate a series of operations all of whose terms are to be affected by the sign preceding the parentheses. Whenever this is done, the operations within a set of parentheses are usually performed separately. This is illustrated in the following example:

Example 6. Find the value of $(f + d) \div b + c(4d - fb + e)$ when $b = 2$, $c = 3$, $d = 4$, $e = 5$, $f = 6$.

Solution: $(f + d) \div b + c(4d - fb + e)$

$$= (6 + 4) \div 2 + 3(4 \cdot 4 - 6 \cdot 2 + 5)$$
$$= (10) \div 2 + 3(16 - 12 + 5)$$
$$= 10 \div 2 + 3(9)$$
$$= 5 + 27$$
$$= 32 \textit{ Ans.}$$

Problems

Evaluate each of the following, using the equivalents $a = 3$, $b = 5$ $c = 7$, $d = 8$, $e = 2$:

1. $16 - (8 - 2)$
2. $6 \div (3 - 2)$
3. $18 \div (2 \cdot 3 - 4 + 2 \cdot 2)$
4. $18 \div (2 + 3 - 4 + 4 \cdot 2)$
5. $(6 - 3)(17 - 2 \cdot 5)$
6. $(10 - 3)(16 - 3 \cdot 2 + 8)$
7. $(a + e)(c + d)$
8. $ab(a + b)$
9. $(a + c)(b + d)$
10. $a + c(a + c)$
11. $a(b + c) + de$
12. $a(2b + c) - (6c + 6b) \div 3a$
13. $c(4d - 5b) - ad$
14. $(2b - a)(2d - 2a + 4e)$
15. $bd \div (ab - 2c + ad)$
16. $(2b + 8c) \div (9d - bc - 2e)$
17. $(6c - ab) \div (10e + ac - 2ed)$
18. $(a + b) \div d + e(c + d)$
19. $(6d - 4b) \div c + e(3a + 2b)$
20. $(0.65 + 1.45)(3.67 - 2.66)$

CHAPTER II

ADDITION OF SIGNED NUMBERS

6. Signed Numbers.—In arithmetic we deal only with positive numbers, but in algebra we use positive and negative numbers. A number which has no sign preceding it is always a positive number, but the plus sign is often used to emphasize the fact. A negative number must always be preceded by a minus sign. When plus and minus signs are thus used to designate the kind of number with which we are dealing, they are called *signs of quality*, and the numbers with which they are associated are known as *signed numbers*.

$+5$ is read "positive five"; -5 is read "negative five."

The *absolute value* of a number is the number without its sign. For example, 8 is the absolute value of $+8$ and, also, of -8.

7. Addition of Signed Numbers.—In order to become proficient in adding positive and negative numbers, the student should memorize the rules here given and always keep them in mind when adding.

Rule 1.—*To add two numbers having like signs, find the sum of their absolute values and place their common sign before the answer.*

Rule 2.—*To add two numbers having unlike signs, find the difference of their absolute values and place the sign of the larger number before the answer.*

When adding more than two numbers, it is usually best to find the sum of all the positive numbers, then the sum of all of the negative numbers, and then add these two sums as explained in Rule 2.

Example 1. Add the following:

(a)	(b)	(c)	(d)
$+7$	7	-7	$-7x$
$+4$	-4	4	$-4x$

Solution:

(*a*) By Rule 1, a positive 7 plus a positive 4 gives 11.
(*b*) By Rule 2, a positive 7 plus a negative 4 gives 3.
(*c*) By Rule 2, a negative 7 plus a positive 4 gives -3.
(*d*) By Rule 1, a negative $7x$ plus a negative $4x$ gives $-11x$.

The solution may be arranged in either of the two following forms:

(*c*) $-7 + (+4) = -3$ *Ans.*, or (*c*) $\begin{array}{r} -7 \\ 4 \\ \hline -3 \end{array}$ *Ans.*

Problems

Add the following:

1. $\begin{array}{r} +7 \\ -3 \\ \hline \end{array}$

2. $\begin{array}{r} -6 \\ +5 \\ \hline \end{array}$

3. $\begin{array}{r} +7 \\ +2 \\ \hline \end{array}$

4. $\begin{array}{r} +\ 8 \\ -15 \\ \hline \end{array}$

5. $\begin{array}{r} -10 \\ 15 \\ \hline \end{array}$

6. $\begin{array}{r} 10 \\ -26 \\ \hline \end{array}$

7. $\begin{array}{r} -32 \\ -27 \\ \hline \end{array}$

8. $\begin{array}{r} 9 \\ -25 \\ \hline \end{array}$

9. $\begin{array}{r} -12 \\ -72 \\ \hline \end{array}$

10. $\begin{array}{r} -3R \\ 7R \\ \hline \end{array}$

11. $\begin{array}{r} 8x \\ -4x \\ \hline \end{array}$

12. $\begin{array}{r} -10x \\ -\ \ 7x \\ \hline \end{array}$

13. $\begin{array}{r} -14n \\ 21n \\ \hline \end{array}$

14. $\begin{array}{r} 5xy \\ -10xy \\ \hline \end{array}$

15. $\begin{array}{r} 16abc \\ -10abc \\ \hline \end{array}$

16. $\begin{array}{r} -22k \\ 39k \\ \hline \end{array}$

17. $\begin{array}{r} -32s \\ 15s \\ \hline \end{array}$

18. $\begin{array}{r} -27t \\ -33t \\ \hline \end{array}$

19. $\begin{array}{r} 20w \\ -18w \\ \hline \end{array}$

20. $\begin{array}{r} -10m \\ 10m \\ \hline \end{array}$

21. $\begin{array}{r} 25xyz \\ -14xyz \\ \hline \end{array}$

In the following problems, addition is indicated by a plus sign between two sets of parentheses. Find the value of the indicated sums.

22. $(+7) + (-10)$
23. $(+6) + (+8)$
24. $(-8) + (-12)$
25. $(-22) + (10)$
26. $(-25s) + (-32s)$
27. $(38n) + (-18n)$
28. $(38n) + (+18n)$
29. $(-27x) + (17x)$
30. $(-35y) + (-15y)$
31. $(50) + (-40)$
32. $(49x) + (-20x)$
33. $(-32R) + (-22R)$
34. $(-21a) + (21a)$
35. $(15b) + (-15b)$
36. $(58c) + (-28c)$
37. $(-51k) + (-24k)$
38. $(-60t) + (20t)$
39. $(37w) + (-17w)$

8. Adding three or more numbers is accomplished as explained in the last large paragraph of Sec. 7.

Example 2. Add the following:

$$\begin{array}{r} -\ 6n \\ 5n \\ 12n \\ -23n \\ \underline{-15n} \end{array}$$

Solution:

$$\begin{array}{r} -\ 6n \\ 5n \\ 12n \\ -23n \\ \underline{-15n} \\ -27n\ \textit{Ans.} \end{array}$$

Adding the two positive numbers $5n$ and $12n$ gives $17n$.
Adding the three negative numbers $-23n$, $-6n$, and $-15n$ gives $-44n$.
$(17n) + (-44n) = -27n$

Add the following:

1.	**2.**	**3.**	**4.**	**5.**
+10	+ 8	− 7	−10 books	+ 6 ft.
− 5	− 4	+ 5	+ 4 books	− 5 ft.
− 4	+ 6	− 3	+12 books	− 8 ft.
+ 9	− 5	+ 4	+20 books	−12 ft.
− 3	+18	−20	− 4 books	+ 9 ft.
+11	− 4	−71	− 8 books	+20 ft.

6.	**7.**	**8.**	**9.**	**10.**
− 4 in.	−20 cts.	− 6x	5n	15y
8 in.	5 cts.	4x	7n	−21y
− 2 in.	− 6 cts.	−10x	−10n	−16y
8 in.	12 cts.	−15x	8n	−10y
−10 in.	−36 cts.	−20x	4n	2y

11.	**12.**	**13.**	**14.**	**15.**
16	25t	20d	8r	− 2s
−12	−12t	32d	−10r	17s
−14	9t	−17d	− 5r	−11s
8	− 6t	5d	− 3r	12s
4	3t	−15d	10r	− 6s

Find the value of the sums indicated in the following:

16. $-12 + (-5) + (+10) + (-6)$
17. $-6 + (-4) + (+5) + (-7) + (+5)$
18. $5 + (+12) + (-16) + (-1)$
19. $14 + (+10) + (+6) + (-20)$
20. $7 + (-2) + (-5) + 6 + (-10)$
21. $-10 + (-16) + (-4) + 8$
22. $-15 + (-11) + 12 + (-7)$
23. $-25 + (-15) + (-27) + (-10)$

Example 3. Add $7b + 19 - 12b + 5b - 32 + 8b + 4$.

Solution: Arrange the numbers containing the letter b in one column and those which do not contain b in a second column. Then add the numbers in each column.

$$\begin{array}{rr} 7b & +\ 19 \\ -12b & -\ 32 \\ 5b & +\ \ 4 \\ 8b & \\ \hline 8b & -\ \ 9 \end{array} \quad \textit{Ans.}$$

Add the following:

24. $6a + 5a + 10 - 18$

25. $6x - 3 + 5x - 2 - 11x - 12x - 8$

26. $8n - 5 + 4n - 9 + 10n - 27$

27. $5t - 8 - 6t + 9 - 10t - 11 + 25t$

28. $18 + 104 - 27y - 14 + 15 - 22y - 18 + 25$

29. $-14c + 26 - 31 + 8c - 15 - 13c + 26c$

30. $37r + 24r - 45 - 50r - 25 + 70$

9. Terms.—A term is a part of an algebraic expression included between two signs $+$ or $-$.

Example. $6a + 5bc - 4a^2c$

$6a$, $5bc$, and $-4a^2c$ are terms of the expression $6a + 5bc - 4a^2c$.

When the expression contains a fraction or parentheses, the above definition must be modified. For example, in the expression $5a - \frac{a+b}{a-b} - 7(a - 2b - c)$, there are only three terms, namely, $5a$, $-\frac{a+b}{a-b}$, $-7(a - 2b - c)$. This is true because the fraction and the expression within the parentheses are treated as units.

10. Monomials.—A monomial is a term which does not contain a plus or minus sign except to indicate whether the monomial is positive or negative.

Example. $+5$, $-7x^2y^3$, $-8ab^2c$ are all monomials.

11. Binomials.—A binomial is an algebraic expression of two terms.

Example. $4x + 2y$; $3a^2b - 4c^2d^2$

12. Trinomial. Polynomial.—A trinomial is an algebraic expression of three terms. A polynomial is an algebraic expression of two or more terms.

13. Exponent.—An exponent is a small figure or letter written to the right and a little above a quantity to indicate how many times the quantity is to be used as a factor.

Example. 7^4 means $7 \times 7 \times 7 \times 7$.
x^3 means $x(x)(x)$.

14. Power.—A power of a number is the result obtained by multiplying that number by itself a definite number of times.

Example. 27 is the third power of 3. $3^3 = 27$

15. Similar Terms.—Similar terms are terms which contain the same letters, and the letters must carry the same exponents.

Example. $4x^2y$, $3x^2y$, $5x^2y$ are all similar terms.
$3xy$, $5x^2y$, $7xy^2$ are all dissimilar terms.

We cannot add 4 apples and 6 pears, but we can only indicate the addition as follows: 4 apples + 6 pears.

We can add like quantities, thus: 4 apples + 6 apples = 10 apples.

In the same way, we can add or subtract only algebraic terms which are similar, although the fact that dissimilar terms are to be added or subtracted can be indicated by using plus and minus signs.

Example 4. Add the polynomials $12a - 7b + 18c$, $-25a + 19b - 22c$ and $-21a + 30b$.

Solution: Arrange the similar terms in columns; then add.

$$\begin{array}{rrr}
12a & -\ 7b & +\ 18c \\
-25a & +\ 19b & -\ 22c \\
-21a & +\ 30b & \\
\hline
-34a & +\ 42b & -\ 4c
\end{array}\ \textit{Ans.}$$

Example 5. Add the polynomials $19r^2 + 36r - 24$, $12r + 9$, $-8r^2 - 16$, $-29r^2 - 40r + 36$.

Solution:

$$\begin{array}{rrr}
19r^2 & +\ 36r & -\ 24 \\
 & +\ 12r & +\ 9 \\
-\ 8r^2 & & -\ 16 \\
-29r^2 & -\ 40r & +\ 36 \\
\hline
-18r^2 & +\ 8r & +\ 5
\end{array}\ \textit{Ans.}$$

Problems

Add the following expressions:

1.
$$\begin{array}{r} 3x^2 - 7x + 8 \\ -2x^2 + 4x - 2 \\ 3x^2 - 2x - 12 \end{array}$$

2.
$$\begin{array}{r} 4x^2 - 9x + 14 \\ -7x^2 - 4x - 27 \\ \underline{-4x^2 + 3x - 8} \end{array}$$

3.
$$\begin{array}{r} 18r^2 - 9rn + 17n^2 \\ 3r^2 + 4rn - 7n^2 \\ \underline{-4r^2 - 3rn + 4n^2} \end{array}$$

4.
$$\begin{array}{r} 0.6a + 0.5b - 0.9c \\ -0.4a - 0.7b - 0.5c \\ \underline{0.1a + b - 0.2c} \end{array}$$

5.
$$\begin{array}{r} 1.7m - 2.5r - 8.1s \\ -2.5m \qquad + 5.6s \\ + 9.2r - 7.5s \\ \underline{3.8m - 2.9r - 3.7s} \end{array}$$

6.
$$\begin{array}{r} 2.1x - 5.2y + 8.6z \\ x - y - z \\ -0.6x + 2.1y - 0.5z \\ \underline{-1.2x + 4.1y - 3.0z} \end{array}$$

7.
$$\begin{array}{r} -9x + 4y - 8z \\ -2x - 2y - 7z \\ 7x - 3y + 5z \end{array}$$

8.
$$\begin{array}{r} 7y^3 - 2y + 4 \\ y^3 \qquad - 2 \\ \underline{-4y^3 - 3y + 10} \end{array}$$

9.
$$\begin{array}{r} 7x - 4xy + 5y \\ -2x + 8xy \qquad \\ \underline{6x - 8xy - 2y} \end{array}$$

10.
$$\begin{array}{r} 3.5c - 2.9d - 0.7e \\ -0.9c - 5.4d - 7.2e \\ \underline{-2.1c + 3.8d + 2.5e} \end{array}$$

11.
$$\begin{array}{r} -4.3h - 2.7k - 9.9n \\ + 5.8k - 2.4n \\ -3.7h - 4.5k + 5.7n \\ \underline{10h \qquad - 1.2n} \end{array}$$

12.
$$\begin{array}{r} 5w - 0.3x + 1.7y \\ 0.2w + x - y \\ -2w + 3x - 2y \\ \underline{-w - 2.1x + 4.2y} \end{array}$$

13. $6a^2 - 3a - 12 + 75a^2 - 140 + 67a - 19a^2$

14. $7x^3 + 145 - 15x^3 - 8x^2 - 190 + 95x^3 - 138$

15. $9a^2 + 3ab + 4b^2 - 7ab - 8a^2 + 5b^2 - 10ab + 8b^2$

16. $6c^2 - 5c + 8 - 10c - 28 + 12c^2 - 18c + 25c^2 - 36 + 17c - 35c^2 + 37c - 35 + 12c^2 - 8c - 20c^2$

17. $9.5x^2 - 1.5 - 4.8x^2 - 7.4x + 36.4 - x^2 - x + 12 - 5.8x + 9.7x^2 - 32 + 1.7x^2 - 3.6x + 7.5$

18. $0.07x - 1.05 + x^2 - 3.25 + 1.19x - 1.08x^2 + 7.52 - 13.05 + 2.27x^2 - 5.82 - 2.36x - 10.46x^2 + 7.5$

19. $3.05x^2 - 14 + x - 4x^2 - 3.89 - 7.67x^2 + 4.75 - 7.65 + 19.25 + 18.67x^2 + 10.65x + 7.11$

20. $1.07a - 3.72 + a^2 + 4.28 - 2.35a + 7.69 - 4.72a - 1.68 + 5.76a + 3a^2 - 2.75 + 4.19a$

CHAPTER III

SOLUTION OF SIMPLE EQUATIONS

16. An *equation* is a mathematical statement which expresses the equality of two or more quantities. For example, the statement $4 + 5 = 9$ is an equation. It expresses in mathematical language the fact that 4 plus 5 is equal to 9.

Complete each of the following equations:

1. 7 desks + 3 chairs + 2 desks + 5 chairs = ? desks + ? chairs
2. 8 nails + 5 tops − 3 tops + 4 nails = ? nails + ? tops
3. 9 pens + 8 pencils − 5 pens − 3 pencils = ? pens + ? pencils
4. $3x + 5x = ?\,x$
5. $2x + 6x + 4x = ?\,x$
6. $5x + 9x - 3x - 7x = ?\,x$
7. $5b + 4c + 3b - 2c = ?\,b + ?\,c$
8. $6b + 3x + 4b + 8x = ?\,b + ?\,x$

17. An equation is always divided into two parts by the sign of equality. These two parts of the equation are called the *members* of the equation or the two *sides* of the equation. We speak of them as the "right" and "left" sides of the equation, depending upon whether the quantities to which we have reference are on the right-hand or the left-hand side of the equal sign.

It is always necessary that we keep our equations balanced. If you place two unequal weights one on each pan of a common scale, the scale will not balance. In the same way, our equation will not balance if the quantities on each side of it are unequal. We may make any changes we like in the equation provided that we do not destroy its balance. The following illustrations will help to make this clear to the student.

$4 + 5 = 9$. Suppose that we add 4 to each side of the equation. It will then read $4 + 5 + 4 = 9 + 4$, and the two

sides still balance. Similarly, if we subtract the same number from each side, the equation will still balance.

Now let us multiply each side of the equation by 4. We have then $16 + 20 = 36$. The equation still balances.

Now divide the equation by 4, and we obtain the result $1 + 1\frac{1}{4} = 2\frac{1}{4}$. Again the resulting equation balances.

We can now state the rule that whatever is done to one side of an equation must be done to the other side. We can add the same number or equal numbers to both sides of an equation. We can subtract the same number or equal numbers from each side of the equation. We can multiply or divide both sides of the equation by the same number. We cannot add to one side and subtract from the other, neither can we multiply one side and divide the other without destroying the balance of the equation.

18. Solving an Equation.—By solving an equation is meant the process of determining the value of the unknown number or numbers in the equation.

Example 1. $x + 5 = 12$. Find the value of x.

Solution:

$$\begin{array}{rcll} x + 5 & = & 12 & \text{subtracting 5 from each side} \\ -5 & = & -5 & \\ \hline x + 0 & = & 7 & \\ x & = & 7 & \end{array}$$

19. Checking an Equation.—To check the answer obtained from the solution of an equation, substitute the answer obtained for the unknown quantity throughout the original equation.

In Sec. 18, the solution of the equation $x + 5 = 12$ is found to be $x = 7$. This result is checked as follows:

Check:

$$\begin{array}{rcll} x + 5 & = & 12 & \\ 7 + 5 & = & 12 & \text{by substituting 7 for } x \\ 12 & = & 12 & \text{Results agree.} \end{array}$$

Since $7 + 5$ does equal 12, x must have a value of 7 in the given equation.

Problems

Solve and check each of the following equations by the method illustrated in Sec. 18 and 19:

1. $x + 2 = 10$	**5.** $y + 12 = 26$	**9.** $t - 4 = 12$
2. $y - 6 = 12$	**6.** $w - 4 = 2$	**10.** $w - 2 = 8$
3. $z + 4 = 8$	**7.** $s - 12 = 20$	**11.** $y + 7 = 15$
4. $x - 7 = 1$	**8.** $r + 5 = 19$	**12.** $x - 4 = 1$

20. Transposition.—By transposition is meant the process of moving a quantity from one side of an equation to the other, without performing the detailed operation of addition or subtraction. The signs + and − are used to indicate positive and negative numbers as well as to indicate addition and subtraction, as explained in Chap. II, Sec. 6. When transposing a term, its sign must be changed when it is moved to the opposite side of the equation. Changing signs means changing a plus sign to minus or a minus sign to plus. By transposing, we obtain the same result as would be obtained if we added to or subtracted from each side of the equation, as is done in Sec. 18, but we do it in a quicker and more convenient way.

Example 2. Solve the equation $x - 7 = 14$.

Solution: $x - 7 = 14$

$x = 14 + 7$ by transposing the 7 and changing its sign

$x = 21$ *Ans.*

Check: $x - 7 = 14$

$21 - 7 = 14$ by substituting 21 for x

$14 = 14$

Example 3. Solve and check the equation $5 = 12 - y$.

Solution: When the right-hand member of the equation contains one or more terms containing the unknown quantity, bring all unknown numbers to the left side of the equation and all known numbers to the right. Change the signs of all terms when transposing.

$5 = 12 - y$

$y = 12 - 5$ by transposing the y and the 5

$y = 7$ *Ans.*

Check: $5 = 12 - y$

$5 = 12 - 7$

$5 = 5$

Problems

Solve each of the following equations by transposing terms, as illustrated in the two foregoing examples. Check your answers.

1. $x + 5 = 19$	11. $k + 12 = 4$	21. $17 = 27 - n$
2. $y - 4 = 12$	12. $12 = 18 - x$	22. $31 = 45 - x$
3. $5 = 14 - x$	13. $15 = 27 - y$	23. $4 = 23 - y$
4. $6 = 21 - w$	14. $z + 9 = 25$	24. $w - 9 = 4$
5. $t - 8 = 2$	15. $x - 6 = 12$	25. $w + 9 = 4$
6. $r + 9 = 27$	16. $w - 8 = 4$	26. $n - 18 = 7$
7. $s + 11 = 12$	17. $n + 5 = 7$	27. $n + 18 = 7$
8. $8 = 21 - x$	18. $n - 9 = 2$	28. $15 = 24 - n$
9. $x + 14 = 5$	19. $m + 8 = 6$	29. $15 = 7 - x$
10. $y + 20 = 9$	20. $m + 9 = 2$	30. $21 = 6 - y$

21. Coefficient.—Any factor of a term may be considered to be the coefficient of the other factor of the term. For example, in the term $6cd$, 6 is the coefficient of cd, $6c$ is the coefficient of d, d is the coefficient of $6c$, etc.

Example 4. Solve the equation $5n = 35$.

Solution: To solve equations of this type, divide both terms of the equation by the coefficient of the unknown quantity. In this case, we divide the equation by 5, which is the coefficient of n.

$$5n = 35$$
$$n = 7 \text{ } Ans.$$

Check:

$$5n = 35$$
$$5(7) = 35$$
$$35 = 35$$

Problems

Solve and check each of the following equations:

1. $3x = 18$	7. $0.2t = 10$	13. $39n = 13$
2. $10s = 40$	8. $0.01r = 5$	14. $46m = 23$
3. $6y = 24$	9. $0.075s = 1.5$	15. $15k = 6$
4. $2t = 18$	10. $0.06m = 3.6$	16. $25h = 6.25$
5. $3y = 48$	11. $0.3n = 3.3$	17. $50x = 7.5$
6. $11z = 33$	12. $0.08w = 0.12$	18. $40y = 9$

Example 5. If 6 is added to a certain number, the result is 21. What is the number?

Solution: Write an equation from the conditions given in the problem and solve the equation.

Represent the number by n.
Then $6 + n$ represents the number with 6 added to it.
We can, therefore, write the equation

$$6 + n = 21$$
$$n = 21 - 6$$
$$n = 15 \textit{ Ans.}$$

Example 6. The product of 4 and an unknown number is 36. What is the unknown number?

Solution: Let x = the number
Then $4x$ = four times the number

$$4x = 36$$
$$x = 9 \textit{ Ans.}$$

Problems

Solve each of the following problems, using the algebraic method illustrated in Ex. 5 and 6. The student should become familiar with the method of representing unknown quantities by means of letters and should be able to obtain an equation from the conditions stated in the problem.

1. A certain number when multiplied by 5 is equal to 40. Find the number.

2. If 15 is added to a number, the resulting sum is 38. What is the number?

3. The product of a certain number and 12 is 108. What is the number?

4. A man is 5 times as old as his son. How old is the son if his father is 45 years old?

5. John is 6 years older than his friend George. If George is 14 years old, how old is John?

6. 0.2 of a certain number is 5. What is the number?

7. A boy has a certain sum of money in his bank. If he had 35 cents more, he would have \$1. How much money has he?

8. If 0.6 is added to a number, the sum is 2.7. What is the number?

9. If 1.5 is subtracted from a number, the result is 3.2. Find the number.

10. When a certain number is multiplied by 0.04, the product is 0.036. What is the number?

11. Twelve years from now, John will be 21 years old. How old is he now?

12. Five times a certain number is 0.35. What is the number?

13. When a number is multiplied by 7, the product is 0.42. Find the number.

14. Nine years ago, George was 15 years old. How old is he now?

15. Eleven years ago, Bertha was 11 years old. How old is she now?

16. Multiply a certain number by 0.3, and the product is 0.225. What is the number?

Example 7. Solve and check the equation $6x + 4x + x = 121$.

Solution: $6x + 4x + x = 121$

$11x = 121$ by adding the unknown terms $6x$, $4x$, and x

$x = 11$ *Ans.*

Check:

$$6x + 4x + x = 121$$
$$6(11) + 4(11) + 11 = 121 \text{ by substituting 11 for } x$$
$$66 + 44 + 11 = 121$$
$$121 = 121$$

Example 8. Solve and check the equation $3x + 4x - 0.9 = 3.3$.

Solution:

$$3x + 4x - 0.9 = 3.3$$
$$3x + 4x = 3.3 + 0.9 \text{ by transposition.}$$
$$7x = 4.2$$
$$x = 0.6 \text{ } Ans.$$

Check:

$$3x + 4x - 0.9 = 3.3$$
$$3(0.6) + 4(0.6) - 0.9 = 3.3$$
$$1.8 + 2.4 - 0.9 = 3.3$$
$$3.3 = 3.3$$

Problems

Solve and check each of the following equations:

1. $6x + 5x = 77$

2. $3x + 7x = 120$

3. $3x + 2x + 7 = 12$

4. $2x + 6x = 4$

5. $9x - 5x - 3 = 17$

6. $8x - 7x - 4 = 8$

7. $12x - 3x + 7x = 48$

8. $3x - 7 + 4x = 42$

9. $5x + 8 - 2x = 56$

10. $12x - 4 - 3x = 77$

11. $1.3x + 5 - 1.1x = 7$

12. $1.4x - 6 + 2.1x = 8$

13. $3.1x - 3.7 - 1.4x = 6.5$

14. $2.4x + 7.6x - 3x = 5.6$

15. $3.8x - 5.9 + 2x = 23.1$

16. $1.6x - 13.2 + 3.1x = 5.6$

Example 9. Solve and check the equation $8x - 3 = 5x + 9$.

Solution: $8x - 3 = 5x + 9$

Bring all of the unknown terms into the left-hand member of the equation and all of the known terms into the right-hand member. This gives

$$8x - 5x = 9 + 3$$
$$3x = 12$$
$$x = 4 \text{ } Ans.$$

Check:

$$8x - 3 = 5x + 9$$
$$32 - 3 = 20 + 9$$
$$29 = 29$$

Example 10. Solve and check the equation $5x - 6 + 3x = 9 + 4x - 3$.

Solution:

$$\begin{aligned} 5x - 6 + 3x &= 9 + 4x - 3 \\ 5x + 3x - 4x &= 9 - 3 + 6 \\ 4x &= 12 \\ x &= 3 \ \textit{Ans.} \end{aligned}$$

Check:

$$\begin{aligned} 5x - 6 + 3x &= 9 + 4x - 3 \\ 15 - 6 + 9 &= 9 + 12 - 3 \\ 18 &= 18 \end{aligned}$$

Problems

Solve and check each of the following equations:

1. $3x - 2 = 10$
2. $2x + 4 = 12$
3. $4x + 11 = 31$
4. $5x - 8 = 37$
5. $9k - 13 = 86$
6. $7r + 5 = 82$
7. $11r - 9 = 134$
8. $3x + 6 = 2x + 11$
9. $6x - 7 = 3x + 5$
10. $3y + 10 = y + 24$
11. $x + 5x - 2x = 64$
12. $5x - 2x + x = 52$
13. $8y - 3y + 5y = 120$
14. $3t - 5 + 6 = 58$
15. $11x - 3x - 13 = 43$
16. $5x + 4 + 7x = 112$
17. $7x - 15 = 153 - 14x$
18. $17x + 13 = 200$
19. $19x - 7 = 240$
20. $13x - 4x = 42 - 5x$
21. $14z - 5z - 77 = 2z$
22. $13h - 198 = 9h - 5h$
23. $15y - 95 - 3y = 7y$
24. $76x - 23x = 180 - 7x$
25. $95y - 497 = 32y - 8y$
26. $5x - 12x + 7 + 11x = 71$
27. $7y - 17 - 15y = 152 - 21y$
28. $4y - 5 + 7y = 116$
29. $5s - 7s = 24 - 8s$
30. $3x - 7 - 5x = 113 - 17x$
31. $5z + 7 + 9z - 5 = 105 + 2$
32. $5y - 9 = 3y + 7$
33. $15x - 37 = 12 + 8x$
34. $3x - 7 - 5x + 10 = 39 - 6x$
35. $5x - 12 = 3x$
36. $7x - 11 = 10$
37. $4z - 5 = 61 - 7z$
38. $50 + 3x - 40 + 2x = 5x + 94 - 2x$
39. $14y + 5 - 7y + 3 = 4y + 95$
40. $12z - 241 = 4z + 48 - 9z$

Oral Exercises

1. A number is 6 greater than b, what is the number? Express a number c greater than b.

2. Express the product of x and y diminished by 7.

Express each of the following statements in the form of an equation:

3. The total cost of 10 articles at r cts. each is c.

4. The total cost of n articles at r cts. each is c.

5. Four times the number x decreased by 5 is equal to 14.

6. 3 times the product of a and b equals c decreased by d.

7. On a dining room table there are a certain number of pieces of silverware. Represent these by x. There are a knives, b forks, and c spoons.

8. In the electrical laboratory there are a certain number of series, shunt, and compound motors represented by s, s', and c, respectively. The total number of motors is m.

9. A boy walks a miles for c days. Altogether he walks x miles.

10. A baseball team played g games; w games were won and l games were lost.

11. A fair was attended by x people, they came in c cars, and the average number of people per car was a.

22. Solution of Problems.—In solving problems, the first essential is that the student read the problem carefully, if necessary several times, until he is sure that he has a correct understanding of the conditions. The terms and conditions of the problem must then be expressed in their correct relation to each other. Note carefully the following:

1. Let x equal one of the quantities which you are asked to find.
2. Represent all other quantities asked for in terms of x.
3. Write an equation from the conditions stated in the problem.
4. Solve the equation for x.
5. Find the value of the other quantities represented in step 2.
6. Check the values found; they must satisfy the conditions of the original problem.

Example 11. In a certain school there are $2\frac{1}{2}$ times as many girls as there are boys. Altogether there are 350 pupils. How many of them are boys and how many are girls?

Solution: Let x = the number of boys

Then

$$\begin{aligned} 2.5x &= \text{the number of girls} \\ x + 2.5x &= 350 \\ 3.5x &= 350 \\ \left.\begin{aligned} x &= 100 \text{ boys} \\ 2.5x &= 250 \text{ girls} \end{aligned}\right\} &\ \textit{Ans.} \end{aligned}$$

Example 12. A, B, and C together spend \$62. A and C each buy a pair of shoes, A spending \$2 more than C. B buys a suit of clothes and pays \$30 more than A and C together paid for their shoes. How much did each spend?

Solution: Let x = cost of C's shoes

Then

$$x + 2 = \text{cost of } A\text{'s shoes}$$

and

$$x + (x + 2) + 30 = \text{cost of } B\text{'s suit}$$

The total expense is represented by $x + (x + 2) + (x + x + 2 + 30)$

$$\begin{aligned} x + x + 2 + x + x + 2 + 30 &= 62 \\ 4x + 34 &= 62 \\ 4x &= 28 \\ \left.\begin{aligned} x &= \$7 \text{ cost of } C\text{'s shoes} \\ x + 2 &= \$9 \text{ cost of } A\text{'s shoes} \\ 2x + 32 &= \$46 \text{ cost of } B\text{'s suit} \end{aligned}\right\} &\ \textit{Ans.} \end{aligned}$$

Problems

1. In a certain village there are four times as many frame houses as there are brick houses. If the village contains a total of 375 houses, how many of each kind are there?

2. Norman has a stick which is three times as long as Fred's. When placed end to end, the sticks cover 24 ft. How long is each?

3. An electrician and his helper had 96 splices to make on a certain job. If the electrician made twice as many splices as his helper, how many did each make?

4. A boy's father is five times as old as his son, and the boy's mother is 22 years older than he is. The sum of their ages is 64 years. How old is each?

5. A generator and a switchboard together cost \$9,100. If the switchboard cost 6 times as much as the generator, find the cost of each.

6. A merchant sold an electric train and an electric fan for \$34. If the train sold for three times as much as the fan, how much did the merchant charge for each?

7. Fred walked 20 yd. farther than John and together they covered 300 yd. How far did each walk?

8. Two boys divide 67 marbles and Walter gets 21 more than Howard. How many does each receive?

9. Divide 367 into two parts one of which is 33 larger than the other.

10. A man leaves \$2,750 to his three sons. Alfred gets \$100 more than Norman, and George gets \$76 more than Norman. How much does each receive?

11. The total gate receipts for a basketball game were \$327. The expenses amounted to \$36. If the home team gets twice as much as the visiting team after the expenses are paid, how much does each receive?

12. A man in selling merchandise for \$10,000 made a profit amounting to $\frac{1}{4}$ of the cost. What was the cost of the merchandise?

13. A garage owner sold a barrel of oil for \$48, making a profit which amounted to $\frac{1}{5}$ of the cost. Find the cost of the oil.

14. A certain number is 16 larger than another number, and the sum of the numbers is 48. What are the numbers?

15. One number is 23 larger than another, and their sum is 67. Find the numbers.

16. The Philadelphia Athletics played 150 games in 1929. They won 58 games more than they lost. How many victories did they have and how many defeats?

17. One number is 12 smaller than another, and the sum of the two is 108. Find the numbers.

18. The difference between two numbers is 13 and their sum is 67. What are the numbers?

19. The difference between two numbers is 38 and their sum is 160. Find the numbers.

20. A man sold his automobile for $1,000, which was at a loss of $\frac{1}{2}$ of the cost. How much did he pay for it?

21. William is 3 times as old as John and Fred is twice as old as John. Alfred is 4 years younger than William. The sum of their ages is 50 years. Find the age of each.

22. A pays $55 more for a motor than B, and C pays $15 less for a generator than A and B together pay for their motors. Altogether they spend $495. How much does each one spend?

23. A basketball team has won 24 more games than it has lost, and another team in the same league has won 18 more games than it has lost. How many games did each win, if together they have played 108 games and each has played the same number of games?

24. The sum of 3 numbers is 153. The first is 46 larger than the second, and the third is 58 larger than the first. Find the numbers.

25. The sum of 3 numbers is 120. The second is 3 times as large as the first, and the third is twice as large as the second. Find the numbers.

CHAPTER IV

OHM'S LAW. SOLUTION OF EQUATIONS

23. The Electric Circuit.—An electric circuit consists of a battery or a generator connected by means of metallic conductors to the apparatus which is to use the electricity. No current of electricity can flow unless there is a complete metallic circuit from one side of the battery or generator to one side of the apparatus using the current and from the other side of this apparatus back to the generator.

There are two kinds or classes of materials from an electrical standpoint: those which will readily permit the flow of an electric current and those which will not permit such a current to flow. These materials are classified as conductors and insulators.

In order to have current flowing through a conductor, an electric pressure is necessary, just as a hydraulic head or water pressure is necessary to cause a flow of water through a system of pipes. This electric pressure is called the "electromotive force" or "voltage" and is usually provided by a battery or a dynamo commonly called a "generator." The function of the connecting wire is to lead the electric current to the point of consumption. The wire, however, has a tendency to resist the flow of current to a certain extent, and this physical property of the wire is called its "resistance."

24. Electrical Units:

The unit of electromotive force is the *volt.*

The unit of current is the *ampere.*

The unit of resistance is the *ohm.*

25. Ohm's Law.—Ohm's law for the electric circuit is usually stated as follows: "The current flowing in a circuit is equal to the pressure divided by the resistance." This law, expressed as a formula in the form in which we shall use it, is

$$E = IR$$

Where E is the electromotive force (e.m.f.) measured in volts.

I is the current measured in amperes.

R is the resistance measured in ohms.

The problems given are to be solved by substituting the proper values in the equation and solving for the unknown term. It is advisable that the student draw a figure illustrating the conditions of each problem. If he does this, solving the problem will mean more to him than mere mechanical substitution in a formula.

Example 1. Find the current flowing in a circuit in which the resistance is 5 ohms and the applied e.m.f. is 90 volts.

Solution: $E = 90$, $R = 5$. We are to find I.

$$E = IR$$

By substituting,

$$90 = I(5)$$

By reversing the equation,

$$5I = 90$$

$$I = 18 \text{ amp. } \textit{Ans.}$$

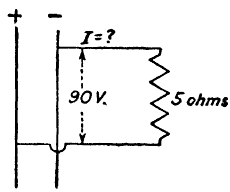


FIG. 1.

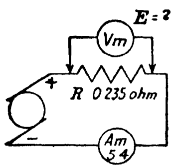


FIG. 2.

Example 2. What will be the reading of a voltmeter which is connected across a 0.235-ohm resistance through which a current of 5.4 amp. is flowing?

Solution: $I = 5.4$, $R = 0.235$

$$E = IR$$

$$E = 5.4(0.235)$$

$$E = 1.269 \text{ volts } \textit{Ans.}$$

Example 3. A voltmeter connected across a certain resistance reads 110 volts. An ammeter connected in series reads 0.35 amp. What is the value of the resistance in ohms?

Solution: $E = 110$, $I = 0.35$

$$E = IR$$

$$110 = 0.35R$$

$$0.35R = 110$$

$$R = 314.3 \text{ ohms } \textit{Ans.}$$

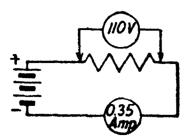


FIG 3.

NOTE.—The answer is given to the nearest tenth.

Problems

1. How much current will flow through a lamp having 18 ohms resistance if the e.m.f. across the lamp is 108 volts?

2. How much current can 20 volts force through 5 ohms?

3. How much current will flow through a 36-ohm resistance if the pressure is 110 volts?

4. How many amperes will a generator furnish to the line if the e.m.f. is 220 volts and the total resistance of line and load is 4 ohms?

5. An electric bell has a resistance of 20 ohms. If it is placed across a storage battery whose voltage is 6.6, how much current flows?

6. The resistance of a tungsten lamp when cold is 30 ohms. How much current flows the instant the circuit is closed, the e.m.f. being 110 volts?

7. If the resistance of the lamp in Prob. 6 rises to 440 ohms when it is at white heat, what will be the steady current on the same circuit?

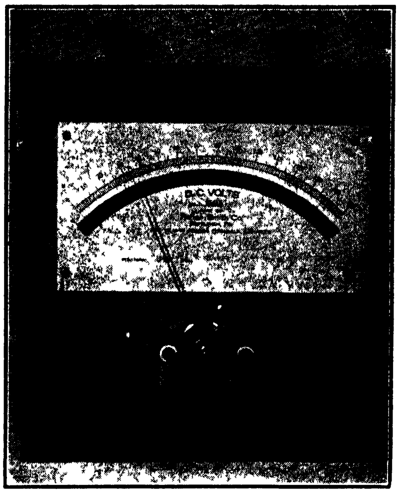


FIG. 4.—Voltmeter with cover removed.

8. 0.35 amp. is needed to ring a small electric bell. If the resistance of the bell is 15 ohms, how many volts are needed to ring it?

9. How many volts will be required to force a current of 1.2 amp. through a resistance of 210 ohms?

10. An electric bulb is supposed to draw 0.25 amp. If its hot resistance is 220 ohms, what is the voltage of the circuit on which it should be used?

11. How much will a voltmeter read if it is connected across a 25-ohm resistance through which a current of 0.075 amp. is flowing?

12. A lamp when connected to a 110-volt circuit draws 0.5 amp. What is the hot resistance of the lamp?

13. An electric flatiron draws 3.3 amp. from a 110-volt line. What is its resistance?

14. Through how many ohms resistance will 550 volts force a current of 25 amp.?

15. A voltmeter, when placed across a piece of apparatus, reads 86 volts. What is the resistance of this piece of apparatus if the current flowing is 1.4 amp.?

16. What must be the resistance of an electric bell if it is to be used on a 5-volt circuit and should draw not more than 0.12 amp.?

17. An ammeter connected in series with a group of lamps reads 18.5 amp. A voltmeter across the lamps reads 107 volts. What is the resistance of the lamps?

18. The resistance of an electric bell is 42 ohms. How many volts are required to ring this bell if the current necessary to ring it properly is 0.3 amp.?

19. What must be the resistance of an electric iron connected to a 112-volt line if it draws 1.4 amp.?

20. What must be the voltage of a line which sends 3.7 amp. through a resistance of 25 ohms?

21. An ammeter whose resistance is 0.007 ohm is designed to carry a maximum of 10 amp. If this ammeter were connected directly across a 110-volt line, how much current would tend to flow?

22. A voltmeter whose resistance is 150,000 ohms is connected across a 120-volt line. How much current flows through the meter?

23. A generator whose voltage is 220 has a 40-ohm resistance connected directly across it. How much current flows through the resistance?

24. Through how great a resistance can a 125-volt generator force a current of 0.25 amp.?

25. If a generator can force 56 amp. through a 10.4-ohm resistance, what is the voltage of the generator?

26. A tungsten lamp whose hot resistance is 220 ohms is supposed to be used on a 112-volt circuit. How much current does it draw from the line?

27. Which resistance is greater: one which requires 16 volts to force a current of 6.1 amp. through it or one which requires 220 volts to force a current of 83.4 amp. through it?

28. Which resistance is greater: one which requires 3 volts to force a current of 0.01 amp. through it or one which requires 8 volts to force a current of 0.35 amp. through it?

29. The current through a 12-ohm resistance is 16.3 amp. What is the voltage across the resistance?

30 How much current will an electric heater whose cold resistance is 47 ohms draw from a 220-volt line the instant the switch is snapped?

31. A voltmeter connected directly across a resistance reads 55 volts. An ammeter in series reads 6.3 amp. How large is the resistance?

32. The plate current of a radio tube is 4 milliamp. when a 5,000-ohm resistance is connected into its plate circuit across a certain "B" battery. What is the voltage drop across this resistance?

NOTE.—1 milliamp. is equal to 0.001 amp.

33. The heater voltage for a 36-type radio tube is 6.3 volts, and its current is 0.3 amp. What is the heater resistance?

34. The voltage across a 50,000-ohm resistance which is connected in series with the plate circuit of a radio tube is 80 volts. What is the current through this resistance (*a*) in amperes, (*b*) in milliamperes?

35. The plate resistance of a 45-type tube is 1,700 ohms. What must be the voltage across the filament and plate of this tube if the plate current is 36 milliamp.?

36. The heater voltage of a 48-type tube is 30 volts, and its hot resistance is 75 ohms. Calculate the heater current in amperes and in milliamperes.

37. What is the filament resistance of a 34-type tube which draws 60 milliamp. when the e.m.f. across its filament is 2 volts?

38. When a 7,000-ohm resistance is connected into the plate circuit of a certain 42-type tube the plate current is 34 milliamp. What is the voltage drop across the 7,000-ohm resistance?

Equations for Review

Solve and check each of the following equations:

1. $14x - 1.2 = 4x - 0.8$
2. $28z - 2.8 = 10z + 8.9$
3. $45x + 17 = 19 - 35x$
4. $18 - 3.5x = 75 - 4.4x + 33$
5. $2.6x - 11 = 2.55x - 10$
6. $39 - 4.73x = 59.4 - 4.9x$
7. $1{,}000x + 2 = 20x + 5.92$
8. $9x - 10 = 11x - 7x + 2$
9. $37x + 35 = 65.76 - 90x - 20.6$
10. $12x - 45 + 10x - 76 = 95 - 15x + 32 + 8x + 6.33$
11. $3x - 31 - 9x - 68 = 32 + 9x + 94 - 18x$
12. $114 - 0.95x + 240 - 0.05x = 2x - 6 - 3.05x + 366$
13. $0.06x - 5x - 92.92 = 3.41x - 12.23x - 58$
14. $9x - 506 + 4.6x - 832.7 - 1.08x = 983.42 - 8.4x$
15. $5.6x + 143.26 + 0.54x - 3.28x = 68.26 - 14.26x + 2.12x + 75.111$

CHAPTER V

SOLUTION OF SERIES CIRCUITS

26. Series Circuits.—In a series circuit the various units comprising the circuit are connected one after another so that the current, starting from one side of the battery or generator, must pass through each unit in turn before reaching the other side of the battery or generator.

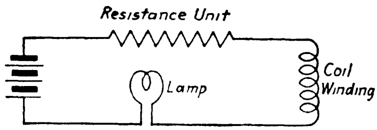


FIG. 5.—Schematic diagram of a resistance unit, a coil and a lamp connected in series with a battery.

The total resistance of a series circuit is equal to the sum of all of the separate resistances. This is expressed by the equation

$$R = a + b + c + \cdots$$

where R is the total resistance of the circuit and a, b, c, etc., are the individual resistances.

There are several important facts regarding series circuits which it will be well for the student to keep in mind. These are as follows:

1. The current in a series circuit is always the same in all parts of the circuit.

2. The total voltage in a series circuit is equal to the sum of the voltages across the different parts of the circuit.

The equation stating this fact is

$$E = E_a + E_b + E_c + \cdots$$

3. The total resistance of a series circuit is equal to the sum of the resistances connected in series.

Ohm's law is used to solve all of the problems in this chapter. Care must be exercised, however, in applying it. For example, if in a series circuit we know the voltage across resistance A and the ohmic resistance of B, we cannot determine the current in the circuit by substituting these two values in Ohm's law and solving for I. Any two values which are used to substitute in Ohm's law must apply to the same part of the circuit.

Example 1. A, B, and C are resistance units connected in series across a 110-volt generator. $A = 12.5$ ohms, $B = 17.9$ ohms, and $C = 5.75$ ohms. Find the current in the circuit and the e.m.f. across each resistance.

Solution: Construct a diagram illustrating the conditions of the problem.

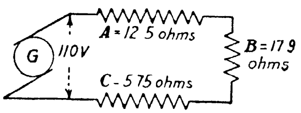


Fig. 6.

$$
\begin{aligned}
R &= a + b + c \\
R &= 12.5 + 17.9 + 5.75 \\
R &= 36.15 \text{ ohms (total resistance)} \\
E &= IR \\
110 &= I(36.15) \\
36.15I &= 110 \\
I &= 3.043 \text{ amp. } \textit{Ans.}
\end{aligned}
$$

Now find the e.m.f. across each resistance:

$$
\begin{aligned}
E &= IR \\
E_A &= 3.043(12.5) \\
&= 38.0375 \\
&= 38.04 \text{ volts}
\end{aligned}
\qquad
\begin{aligned}
E &= IR \\
E_B &= 3.043(17.9) \\
&= 54.4697 \\
&= 54.47 \text{ volts}
\end{aligned}
\qquad
\begin{aligned}
E &= IR \\
E_C &= 3.043(5.75) \\
&= 17.497 + \\
&= 17.50 \text{ volts}
\end{aligned}
$$

Check: $E = 38.04 + 54.47 + 17.5 = 110.01$ volts

Example 2. A, B, and C are connected in series across a 115-volt line. The e.m.f. across A is 27 volts; across B, 45 volts. C has a resistance of 21.5 ohms. Find the current.

Solution:

$$
\begin{aligned}
E &= E_a + E_b + E_c \\
115 &= 27 + 45 + E_c \\
115 - 27 - 45 &= E_c \\
E_c &= 43 \text{ volts} \\
E &= IR \\
43 &= I(21.5) \\
21.5I &= 43 \\
I &= 2 \text{ amp. } \textit{Ans.}
\end{aligned}
$$

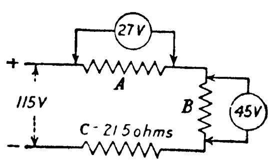


Fig. 7.

This is the current through resistance C. Since we are dealing with a series circuit, it is also the total current through the circuit.

Example 3. A, B, and C are connected in series across a generator. $A = 27.4$ ohms, $B = 38.5$ ohms, $C = 46.9$ ohms. The e.m.f. across C is 70.35 volts. Find the generator voltage.

Solution:

$$\begin{aligned} R &= 27.4 + 38.5 + 46.9 \\ &= 112.8 \text{ ohms} \\ E &= IR \\ 70.35 &= I(46.9) \\ 46.9I &= 70.35 \\ I &= 1.5 \text{ amp.} \\ \text{Generator voltage} &= E = 1.5(112.8) \\ &= 169.2 \text{ volts } Ans. \end{aligned}$$

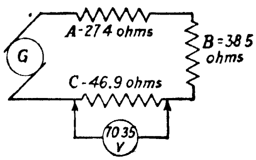


FIG. 8.

Example 4. How much resistance must be connected in series with a 150-ohm relay, which is to be operated from a 24-volt battery, to keep the current through the relay from exceeding 0.026 amp.?

Solution:

$$\begin{aligned} E &= IR \\ 24 &= 0.026R \\ 0.026R &= 24 \\ R &= 923 \text{ ohms} \end{aligned}$$

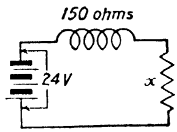


FIG. 9.

This is the value of the total resistance which the circuit must contain.

$$\begin{aligned} R &= a + b + c + \cdots \\ 923 &= 150 + x \\ 923 - 150 &= 773 \text{ ohms } Ans. \end{aligned}$$

27. Degree of Accuracy.—It often happens in actual practice that certain measurements or readings can be accurately made to only two or three figures and that the next figure must be estimated. When making calculations with figures obtained in this way, we must bear in mind the fact that the results obtained cannot be more accurate than the figures upon which they are based.

Suppose that we wish to find the resistance of a coil by the voltmeter-ammeter method. The voltmeter across the coil is read 110.1 volts and the ammeter in series with the coil is read 4.35 amp. In each of these readings, the last figure is estimated. We calculate the resistance of the coil as follows:

$$\begin{aligned} E &= IR \\ 110.1 &= 4.35R \\ 4.35R &= 110.1 \\ R &= \frac{110.1}{4.35} \end{aligned}$$

In performing this division, we should bear in mind the fact that the last figure of each number is approximate. The division is performed below and all of the approximate figures are printed in heavy type.

```
4.35)110.1000(25.31
     870
     2310
     2175
      1350
      1305
        450
        435
```

In stating the result, we should say that the resistance is 25.3 ohms and not 25.31 ohms, because the latter figure would indicate a greater accuracy than is warranted by the facts. In the result 25.31, the 1 is worthless, because it was obtained from figures all of which were approximate, as is seen by examining the division given above.

Another illustration is seen in Ex. 1 of the preceding section. We must assume, of course, that the resistance values given are accurate. In the current value 3.043 amp., the first three figures are accurate and the fourth figure is approximate. In calculating the voltage, we find it to be 38.0375 volts. To leave the answer this way would be misleading, since it was obtained by multiplication from a number in which only the first three figures are exact. We say, therefore, that the answer is approximately 38.04 volts. If in the result 38.0375, the fifth figure had been smaller than 5, we should have dropped the last two figures and the answer would be given as 38.03 volts.

Problems

1. A generator maintains a pressure of 615 volts across a 25-ohm resistance, a lamp of 28 ohms resistance, and a heater of 122 ohms resistance, all connected in series. What is the total resistance, and how much current flows?

2. A coil whose resistance is 8.27 ohms is connected in series with a lamp of 2.38 ohms and a heater of 12.25 ohms resistance. If the current flowing is 6.5 amp., what is the voltage of the circuit?

3. Resistances A, B, and C are joined in series. $A = 300$ ohms, $B = 240$ ohms, and $C = 112$ ohms. The current through B is 2.6 amp. What is the voltage across A; across C; across the whole circuit?

4. A coil of 3.61 ohms and a lamp of 25.9 ohms are connected in series with a heater across a 110-volt line. An ammeter in the circuit shows that 1.32 amp. are flowing. What is the resistance of the heater?

5. If the three resistances mentioned in Prob. 3 were to be connected in series across a 110-volt line, how much current would flow?

6. Three lamps of the same size are connected in series across a 220-volt line. What is the resistance of each of them if they draw a current of 0.6 amp.?

7. In Prob. 6, what would be the reading of a voltmeter connected across each of the lamps in turn?

8. Four arc lamps, each having a resistance of 62 ohms, are connected in series across a 1,650-volt generator. If the current through each lamp is 6.6 amp., what is the resistance of the line wires?

9. How much voltage would be used up in sending the current of 6.6 amp. through the line wires of Prob. 8?

10. How large a resistance would you place in series with a 12-ohm bell which is to draw exactly 0.27 amp. from a 24-volt battery?

11. Three resistances A, B, and C are connected in series across the terminals of a 115-volt line. Voltmeter across A reads 20 volts, across B reads 28 volts. C has a resistance of 26 ohms. How much current is flowing in the circuit?

12. In Prob. 11, what is the value of the resistances A and B?

13. Three resistances A, B, and C are connected in series across a generator. An ammeter in the circuit registers 5.3 amp. A voltmeter placed across B registers 14 volts and when placed across C registers 27 volts. A has a resistance of 32 ohms. What is the value of resistances C and B?

14. What is the voltage of the generator in Prob. 13?

15. Three resistances A, B, and C are connected in series across a generator. An ammeter in the circuit registers 3.8 amp. $B = 39$ ohms, $C = 47$ ohms. A voltmeter across A reads 12 volts. What would a voltmeter across B read; across C? What is the generator voltage?

16. Suppose the generator voltage in Prob. 15 to be 220, C having a resistance of 47 ohms, resistances of A and B unknown. A voltmeter placed across A reads 75 volts and one across B reads 92 volts. How much current is flowing in the circuit? Find the value of resistances A and B.

17. Draw a figure for Prob. 13. Place a voltmeter across A registering 49 volts and one across C registering 38 volts. If the ammeter registers

7.4 amp. and B has a resistance of 2 ohms, what is the resistance of A and C?

18. An arc lamp which is designed to operate on a current of 4.5 amp. is to be used on a 110-volt circuit. If the resistance of the lamp is 10.4 ohms, how much resistance must be connected in series with the lamp?

19. An arc lamp designed to operate on a current of 6.6 amp. has a resistance of 14 ohms. It is to be used on a 125-volt circuit. How much resistance must be connected in series with it?

20. An arc lamp is burning on a 115-volt circuit and is drawing 6.6 amp. A 5-ohm resistance is connected in series with the lamp. What is the resistance of the lamp under these conditions?

21. Three resistances of 27.4 ohms, 39.5 ohms, and 14.2 ohms are connected in series across a 115-volt generator. What must be the resistance of the connecting wires if the current is 1.41 amp.?

22. How much resistance must be connected in series with a 48-ohm bell which is to be operated from a 6-volt battery in order to keep the current from exceeding 0.05 amp.?

23. Three resistances of 17.4 ohms, 11.2 ohms, and 29.3 ohms are connected in series across a 65-volt generator. What must be the resistance of the connecting wires if the current is 1.12 amp.?

In each of the following problems, A, B, and C represent three resistances connected in series:

24. A, B, and C are connected across a 115-volt generator. The e.m.f. across A is 45 volts, across B is 26 volts. C has a resistance of 56 ohms. Find the current in the circuit.

25. The e.m.f. across A is 40 volts and across C is 60 volts. The resistance of B is 32 ohms. The total e.m.f. of the circuit is 120 volts. Find the current.

26. A voltmeter across A reads 72 volts, one across B reads 43 volts. C has a resistance of 17 ohms. An ammeter in the circuit reads 2.5 amp. Find the total e.m.f. across the circuit.

27. The e.m.f. across A is 50 volts, across B it is 90 volts. The total e.m.f. of the circuit is 220 volts. If the resistance of C is 24 ohms, what is the current in the circuit?

28. The total e.m.f. across a circuit is 150 volts. If the voltage across B is 45 volts and across C 70 volts, what is the current in the circuit, the resistance of A being 5 ohms?

29. The e.m.f. across A is 35 volts and across C is 72 volts. Resistance of B is 15 ohms. The total e.m.f. of the circuit is 135 volts. Find the current.

30. A voltmeter across A reads 60 volts, one across B reads 52 volts. C has a resistance of 18 ohms. An ammeter in the circuit reads 3.5 amp. Find the total e.m.f. across the circuit.

Radio Problems

31. When a resistor is connected between the cathode of a radio tube and the negative terminal of its "B" battery, the effect is to produce a negative voltage, or grid bias, on the grid of the tube. When the plate current which passes through a bias resistor is 30 milliamp., how large must the resistor be in order to produce a grid bias of 16.5 volts? (1 milliamp. equals 0.001 amp.)

32. Two 45-type tubes have their filaments connected in series across a battery whose e.m.f. is 6.2 volts. The filament resistance of each tube is 1.67 ohms. How large a resistor must be placed in series with the battery and these tubes to give a filament current of 1.5 amp.?

33. A 25,000-ohm resistance is connected in series with a "B" battery of 250 volts across the plate and filament terminals of a radio tube. If the resulting plate current is 3.5 milliamp., what would a voltmeter read when connected across the same two terminals of the tube?

34. A negative grid bias of 56 volts is to be produced on a radio tube whose plate current is 36 milliamp. How large a bias resistor must be used?

35. A battery of 3 cells is used to furnish the filament current for three 34-type tubes connected in series. A resistor also is connected in series with the battery and the tubes. With the normal filament current of 60 milliamp. flowing, the voltages of the 3 cells are 2.15, 2.03 and 2.09 volts, respectively. The e.m.f. across each tube is 2 volts. What is the e.m.f. across the resistor and what is its resistance value in ohms?

36. Two 27-type tubes are to be connected across a 3-cell battery as described in Prob. 35. With a filament current of 1.75 amp. flowing, the voltages across the cells of the battery are 2.01, 1.95, and 2.07 volts, respectively. If the e.m.f. across the filaments of each of the tubes is 2.5 volts, what is the resistance value of the series resistor?

37. A single 112-A tube is connected across a 3-cell battery as described in Prob. 35, and the current is found to be 250 milliamp. with 5 volts across the tube. If the cell voltages are 1.85, 2.03, and 2.12 volts, what is the ohmic value of the resistor?

38. A 22,500-ohm resistance is connected in series with a 250-volt "B" battery across the plate and filament terminals of a radio tube. A voltmeter connected across the negative terminal of the battery and the plate terminal of the tube reads 179.25 volts. What is the plate current in milliamperes?

CHAPTER VI

EQUATIONS CONTAINING FRACTIONS

28. Fractional Equations.—An equation in which the unknown appears in a fraction is called a "fractional equation."

Example 1. $\frac{y}{8} = 3.$ Find the value of y.

Solution: $\frac{y}{8} = 3.$ Multiply each term by 8, the denominator of the fraction.

$$\frac{y(8)}{8} = 3(8)$$

$$y = 24 \text{ Ans.}$$

Check:

$$\frac{24}{8} = 3$$

$$3 = 3$$

Solve and check each of the following equations:

1. $\frac{x}{2} = 1$	5. $\frac{3x}{2} = 4$	9. $\frac{x}{2} - 5 = 0$	13. $\frac{2x}{3} - 8 = 0$
2. $\frac{x}{3} = 2$	6. $\frac{2x}{3} = 3$	10. $\frac{x}{4} - 7 = 0$	14. $\frac{3x}{4} - 3 = 0$
3. $\frac{x}{4} = 1$	7. $\frac{2x}{5} = 10$	11. $\frac{x}{3} - 2 = 0$	15. $\frac{5x}{3} - 10 = 0$
4. $\frac{x}{2} = 6$	8. $\frac{3x}{4} = 9$	12. $\frac{x}{5} - 1 = 0$	16. $\frac{4x}{17} - 12 = 0$

29. Clearing of Fractions.—When an equation contains several fractions, the simplest method of solving is to eliminate them by a process called "clearing of fractions." To do this, find the least common multiple of the denominators, which we shall term the "Least Common Denominator (L.C.D.)," and multiply every term in the equation by this number.

Example 2. Find the value of x in $\frac{3x}{4} - 6 = \frac{2x}{6} - 1.$

Solution:
$$\frac{3x}{4} - 6 = \frac{2x}{6} - 1$$

multiplying by 12, the L.C.D. gives

$$\frac{3x(12)}{4} - 6(12) = \frac{2x(12)}{6} - 1(12)$$
$$9x - 72 = 4x - 12$$
$$5x = 60$$
$$x = 12$$

Check:
$$\frac{3x}{4} - 6 = \frac{2x}{6} - 1$$
$$\frac{3(12)}{4} - 6 = \frac{2(12)}{6} - 1$$
$$9 - 6 = 4 - 1$$
$$3 = 3$$

30. Principle of Cross-multiplication.—When an equation consists of only two fractions, the process of clearing of fractions in the usual way gives the same result which would be obtained if we multiplied the numerator of each fraction by the denominator of the other. This is called "cross-multiplying."

Example 3. $\frac{4x}{5} = \frac{7}{3}$.

$12x = 35$ by cross-multiplying

$x = {}^{35}\!/_{12} = 2{}^{11}\!/_{12}$

Solve and check each of the following equations:

1. $\frac{x}{2} = \frac{7}{8}$
2. $\frac{x}{3} = \frac{9}{5}$
3. $\frac{y}{4} = \frac{3}{4}$
4. $\frac{3x}{5} - \frac{4}{5} = 7$
5. $5 = \frac{15}{x}$
6. $\frac{5x}{6} + \frac{2}{3} = 9$
7. $\frac{y}{3} = 0$
8. $\frac{2x}{5} = \frac{8}{7}$
9. $\frac{y}{4} = \frac{2}{3} - \frac{1}{6}$
10. $\frac{x}{4} = \frac{1}{16}$
11. $2n + \frac{n}{3} = 14$
12. $3x - \frac{x}{2} = 5$
13. $\frac{y}{3} + \frac{y}{3} = 4$
14. $\frac{r}{2} + \frac{r}{7} = 18$
15. $\frac{s}{5} + \frac{s}{6} = 33$
16. $\frac{x}{7} = \frac{x}{8} + 2$
17. $\frac{2y}{3} + \frac{5y}{6} - \frac{y}{3} = 4$
18. $3x - \frac{x}{3} + \frac{7}{2} - \frac{x}{8} = 34$
19. $\frac{2x}{5} = 9 + \frac{x}{7}$
20. $\frac{x}{8} + \frac{2}{3} = 4$
21. $\frac{n}{4} - \frac{1}{3} = \frac{n}{5} + \frac{2}{3}$
22. $r + \frac{5r}{6} = \frac{11}{2}$
23. $3x - \frac{4x}{11} - 10 = 8 + x$
24. $\frac{35}{3x} = \frac{5}{3}$

25. $\frac{3}{2x} + \frac{2}{3} = \frac{1}{8} + \frac{8}{x}$

26. $7x - \frac{5x}{8} - \frac{12}{7} = \frac{131}{8} + \frac{11x}{2} - \frac{47}{7}$

27. $\frac{5x}{4} - 9 + \frac{2x}{3} - \frac{11x}{12} = 14$

28. $\frac{72}{5} - \frac{45}{x} - \frac{16}{x} = \frac{121}{5x} - 14$

29. $\frac{1}{R} = \frac{1}{8} + \frac{1}{5}$

30. $\frac{1}{x} = \frac{1}{6} + \frac{1}{7} + \frac{1}{3}$

31. $\frac{1}{y} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5}$

32. $\frac{1}{5} = \frac{1}{35} + \frac{1}{x} + \frac{1}{7}$

33. $\frac{1}{2} = \frac{1}{3} + \frac{1}{r} + \frac{1}{12}$

34. $\frac{1}{2} = \frac{1}{10} + \frac{1}{y} + \frac{1}{5}$

35. $\frac{1}{6} = \frac{1}{r} + \frac{1}{36} + \frac{1}{12}$

36. $\frac{1}{4} = \frac{1}{8} + \frac{1}{12} + \frac{1}{x}$

31. Resistances in Parallel.—Resistances are said to be connected in parallel or in multiple whenever it is possible for the current to divide so that only a part of the total current passes through each of the resistances. The parallel portions of a circuit are called "paths" or "branches."

As indicated by the arrows in Fig. 10, the total current from the generator divides into two parts, each coil receiving a part of the total current.

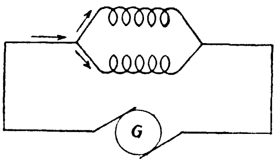


FIG. 10.—Two coils connected in parallel across a generator.

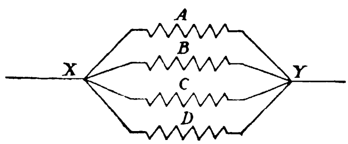


FIG. 11.—Four resistances connected in parallel across the points X and Y.

32. Resistance and Conductance.—The resistance of a conductor is that physical property of the conductor which opposes or limits the current flow. Conductance is just the opposite. It is the ability of a conductor to carry current. The unit of conductance is the *mho*. Conductance may also be defined as the reciprocal of the resistance, that is, one divided by the resistance. Thus, if R is the resistance of a wire, $1/R$ will be the conductance of the wire.

33. Formulas for Resistances in Parallel and Condensers in Series.—When resistances are connected in parallel, each additional resistance so connected provides an additional

path which the current may follow, and, therefore, by increasing the number of resistances connected in parallel, we are increasing the ability of the circuit to carry current. For a parallel circuit therefore, to find the total conductance of the circuit, we add the conductances of the separate branches. This is expressed by the formula

$$\frac{1}{R} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \cdots$$

where R is the total resistance of the circuit, and a, b, c, etc., are the resistances connected in parallel.

To find the combined resistances of a group of parallel resistances, we substitute the value of the individual resistances in the above equation and solve the equation for R.

When several condensers are connected in series, a similar formula applies. The total capacitance C of a group of condensers represented by c_1, c_2, c_3, etc., connected in series, may be determined from the relation

$$\frac{1}{C} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} + \cdots$$

Example 4. Find the total resistance of 11.4 ohms, 6.5 ohms, 8.75 ohms, and 9.37 ohms, connected in parallel.

Solution:
$$\frac{1}{R} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

Substituting,

$$\frac{1}{R} = \frac{1}{11.4} + \frac{1}{6.5} + \frac{1}{8.75} + \frac{1}{9.37}$$

Next, multiply the equation by $11.4R$, that is, the largest of the denominators and the unknown quantity. This gives

$$\frac{11.4R}{R} = \frac{11.4R}{11.4} + \frac{11.4R}{6.5} + \frac{11.4R}{8.75} + \frac{11.4R}{9.37}$$

Now divide each numerator by its denominator. This gives

$$\begin{aligned} 11.4 &= R + 1.754R + 1.303R + 1.217R \\ 11.4 &= 5.274R \\ 5.274R &= 11.4 \\ R &= 2.16 \text{ ohms } \textit{Ans.} \end{aligned}$$

Example 5. Three resistances of 13.6 ohms, 27.45 ohms, and 32.68 ohms are connected in parallel. How large a resistance must be con-

nected in parallel with these three to make the resistance of the group equal to 3.5 ohms?

Solution: In this problem, we know that the total resistance of the parallel group of four resistances is 3.5 ohms. One of the four resistances is unknown. We use the formula for parallel resistances, substitute the values given, and solve for the unknown quantity.

$$\frac{1}{R} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$

$$\frac{1}{3.5} = \frac{1}{13.6} + \frac{1}{27.45} + \frac{1}{32.68} + \frac{1}{d}$$

$$\frac{32.68d}{3.5} = \frac{32.68d}{13.6} + \frac{32.68d}{27.45} + \frac{32.68d}{32.68} + \frac{32.68d}{d}$$

$$9.337d = 2.403d + 1.191d + 1d + 32.68$$

$$9.337d - 2.403d - 1.191d - d = 32.68$$

$$4.773d = 32.68$$

$$d = 6.89 \text{ ohms } Ans.$$

34. Use of Squared Paper.—A sheet of paper whose surface is divided into small squares by equally spaced horizontal and vertical lines is called "squared paper" or "cross-section paper." This kind of paper is very convenient for representing statistics in graphical form and for solving problems graphically. In the next section, we shall see how the problems given in this chapter may be solved graphically.

35. Graphical Solution of Parallel Resistance.—Problems in parallel resistance can be conveniently solved graphically, as illustrated by the two examples following.

Example 6. Find the total resistance of a parallel group of three resistances whose values are 10 ohms, 15 ohms, and 7 ohms.

Graphical solution: Take a sheet of cross-section paper and choose a convenient unit for representing the resistances. In this case, it is convenient to let each square represent a resistance of 1 ohm. Near the left edge of the paper, lay off the line *AB* equal to 10 units so that *AB* represents the 10-ohm resistance. At any convenient point to the right, lay off *CD* equal to 15 units. Draw the lines *AD* and *BC* and mark their point of intersection *E*. The line *EF* represents the resistance of 10 and 15 ohms in parallel. This part of the construction is given in Fig. 12.

In Fig. 13, *EF* has the same length as in Fig. 12, and *HG* represents the 7-ohm resistance. The line *MN* represents the resistance of *EF* and *HG* in parallel. It also represents the resistance of 10 ohms, 15 ohms, and 7 ohms, connected in parallel. Its length is estimated as 3.2 units. The parallel resistance of the group in Ex. 6 is, therefore, 3.2 ohms.

The constructions in Figs. 12 and 13 may be combined in one figure. This is done in the first part of Ex. 7, Fig. 14.

Example 7. Four resistance units, connected in parallel, have a combined resistance of 3.5 ohms. Three of the resistances have the following

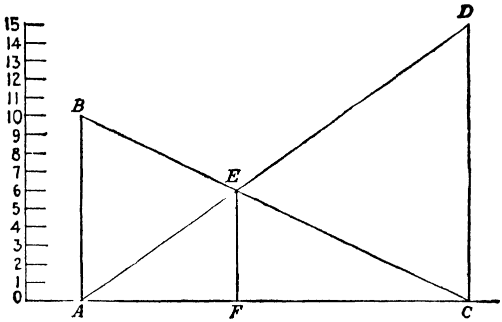


FIG. 12.

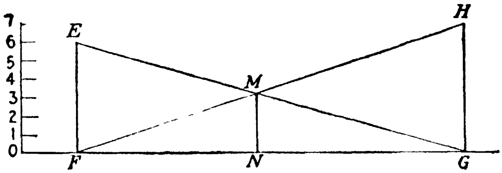


FIG. 13.

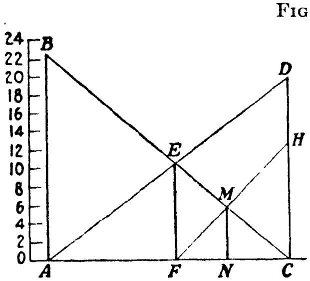


FIG. 14.

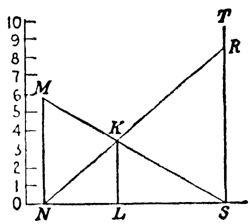


FIG. 15.

values: 19.7 ohms, 12.4 ohms, and 22.5 ohms. What is the resistance of the fourth unit?

Graphical solution: First find the parallel resistance of the three known resistances. This is represented by MN in Fig. 14. Each square represents 2 ohms.

After finding MN, we must determine how large a resistance, connected in parallel with MN, will give a joint resistance of 3.5 ohms. This resistance is found as shown in Fig. 15. The scale used in this figure is

double that of Fig. 14. Lay off MN to scale, and any convenient number of units to the right lay off KL, 3.5 units long. Through the points M and K draw a straight line which meets the base line at S. At S draw ST parallel to MN. Then through the points K and N draw a straight line which intersects ST at R. The line RS represents the resistance unit whose value we desire to know. It measures 8.4 units and the unknown resistance is, therefore, 8.4 ohms.

The graphical method may be used as a convenient check on calculations made by using the formula of Sec. 33.

Problems

1. Three resistances of 3, 6, and 9 ohms are connected in parallel. Find the total resistance of the parallel combination.

2. Four resistances of 4, 8, 12, and 15 ohms are connected in parallel. Find the combined resistance.

3. Three condensers of 6, 8, and 10 mf. are connected in series. Find the total capacitance of the combination.

4. Find the total resistance of 5, 7, and 9 ohms connected in parallel.

5. Find the total capacitance of 18, 24, and 72 mf. connected in series.

6. Find the total resistance of a parallel combination of 10, 12, and 15 ohms.

7. Find the total capacitance of a series combination of 15, 7.5, and 5 mf.

8. Find the total resistance of 6, 8, and 4.8 ohms connected in parallel.

9. Find the total resistance of 9.45, 15.27, and 27.15 ohms connected in parallel.

10. Find the total resistance of 8.25, 6.875, 9.43, and 3.67 ohms connected in parallel.

11. Four condensers of 6.857, 5.46, 4.8, and 7.2 mf. are connected in series. Find the total capacitance.

12. Four resistances of 10.833, 8.67, 13, and 7.8 ohms are connected in parallel. Find the total resistance.

13. Four resistances of 11.57 ohms, 27 ohms, 11.8 ohms, and 6.75 ohms are connected in parallel. Find the total resistance.

14. What voltage would be required to send a total of 15 amp. through a parallel combination of 4.5, 12.7, and 18.6 ohms?

15. How many volts would be required to force a total of 27 amp. through a parallel group of 9.6, 12.9, and 36.5 ohms?

16. How many volts would be required to force 10 amp. through a parallel group of 6.8, 12.75, 8.5, and 1.92 ohms?

17. How many volts would be required to force 30 amp. through a parallel group of 5.63, 8.57, 3, and 13.84 ohms?

18. How many volts would be required to force 20 amp. through a parallel combination of 5.833, 23.33, 6.125, and 4.143 ohms?

In each of the following problems, a, b, c, and d are resistance units connected in parallel, or condensers connected in series.

19. The total resistance of a, b, and c is 2 ohms. $a = 7.8$ ohms, $b = 8.67$ ohms. Find the value of c.

20. $a = 7.94$ ohms, $b = 9.27$ ohms, $c = 5.56$ ohms. The combined resistance of a, b, c, and d is 1.5 ohms. Find the resistance of d.

21. The total capacitance of a, b, c, and d is 9 mf. $a = 28.9$ mf., $b = 47.5$ mf., $c = 51.8$ mf. Find the value of d.

22. a, b, c, and d have a total resistance of 4.5 ohms. $a = 11.13$ ohms, $b = 13.6$ ohms, $c = 25.8$ ohms. Find the resistance of d.

23. a, b, c, and d have a total capacitance of 2 mf. $a = 6$ mf., $b = 8$ mf., $c = 12$ mf. Find the capacitance of d.

24. The total resistance of a, b, c, and d is 1 ohm. $b = 2$ ohms, $c = 4$ ohms, $d = 6$ ohms. Find the resistance of a.

25. The total resistance of a, b, c, and d is 0.9375 ohm. $a = 2$ ohms, $b = 3$ ohms, $d = 5$ ohms. Find the resistance of c.

26. The total capacitance of a, b, c, and d is 1.62 mf. $a = 4.05$ mf., $c = 12.15$ mf., $d = 4.05$ mf. Find the value of b.

27. The total resistance of a, b, c, and d is 1.125 ohms. $a = 2.4$ ohms, $b = 3.6$ ohms, $d = 6$ ohms. Find the resistance of c.

28. The total resistance of a, b, c, and d is 8.1 ohms. $b = 32.4$ ohms, $c = 97.2$ ohms, $d = 214.4$ ohms. Find the resistance of a.

29. The total resistance of a, b, c, and d is 16.4 ohms. $a = 65.61$ ohms, $c = 196.83$ ohms, $d = 393.66$ ohms. Find the resistance of b.

30. $a = 350$ ohms, $c = 210$ ohms, $d = 280$ ohms. Find the resistance of b, if the total resistance of a, b, c, and d is 57.5 ohms.

31. Suppose that 110 volts are connected across each of the resistance groups mentioned in Prob. 1 to 5. Find the current flowing through each resistance and the total current supplied to each group.

HINT.—The e.m.f. across each of the resistances is 110 volts.

32. Suppose that 5 amp. are flowing through the largest resistance of each of the groups mentioned in Prob. 6 to 10. Find the current flowing through each of the other resistance units in each group.

33. A, B, and C are three resistance units connected in parallel across a 110-volt line. A has a resistance value of 15 ohms and B of 22 ohms. If the current through C is 1.5 amp., what is its resistance value? Find the current through A and B.

Example 8. A resistance unit of 12 ohms, one of 7.5 ohms, and an unknown resistance are connected in parallel across a 54-volt battery. The current furnished by the battery is 14 amp. What is the resistance value of the unknown unit?

Solution: Draw a diagram of the connections.

We know that the voltage across the unknown resistance is 54 volts. If we knew how much current it is drawing from the battery we could readily find its resistance. The current through the unknown must be

the difference between the total current and the sum of the currents through the other two resistances. Therefore, first find the currents in these two resistances, as follows:

$$\begin{aligned} 54 &= x(12) \\ 12x &= 54 \\ x &= 4.5 \text{ amp.} \end{aligned} \qquad\qquad \begin{aligned} 54 &= y(7.5) \\ 7.5y &= 54 \\ y &= 7.2 \text{ amp.} \end{aligned}$$

I is the total current and z the current through the unknown.

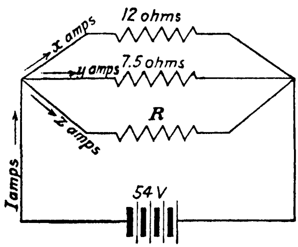


FIG. 16.

$$\begin{aligned} I &= x + y + z \\ \therefore 14 &= 4.5 + 7.2 + z \\ 14 - 4.5 - 7.2 &= z \\ 2.3 &= z \\ z &= 2.3 \text{ amp.} \end{aligned}$$

The resistance R of the unknown is now found by Ohm's law:

$$\begin{aligned} 54 &= 2.3R \\ 2.3R &= 54 \\ R &= 23.5 \text{ ohms } Ans. \end{aligned}$$

34. Three resistances of 6 ohms, 15 ohms, and one of unknown value are connected in parallel across a 105-volt line. The total current flowing is 39.5 amp. Find the value of the unknown resistance.

35. Three resistances of 18 ohms, 25 ohms, and an unknown resistance are connected in parallel across a generator whose e.m.f. is 139.5 volts. Find the resistance of the unknown, if the generator supplies 17.83 amp.

36. Three resistances of 19 ohms, 12 ohms, and an unknown resistance are connected across a 228-volt generator. The total current flowing is 37 amp. Find the value of the unknown resistance.

37. Three resistances A, B, and C are connected in parallel across a generator whose potential is 84 volts. Resistance A measures 12 ohms and B measures 14 ohms. If the generator is delivering 15.8 amp., what is the resistance of C?

38. In Prob. 37, suppose that the brush potential of the generator is 165 volts, resistance of B 11 ohms, and of C 15 ohms. If the total current is 59 amp., find the value of resistance A.

39. In Prob. 37, suppose the brush potential of the generator to be 99 volts, resistance of A to be 22.5 ohms, and of C to be 7.5 ohms. If the generator delivers 23.6 amp., find the value of resistance B.

40. In Prob. 37, suppose the brush potential of the generator to be 63 volts, resistance of A to be 3 ohms, and of B 7 ohms. If the generator delivers 42.6 amp., find the value of resistance C.

41. In Prob. 37, suppose the brush potential of the generator to be 94.5 volts, resistance of A to be 4.5 ohms, and of B 7.5 ohms. If the current through C is 9 amp., find the current delivered by the generator and the resistance of C.

Example 9. Three resistances of 4, 8, and 10 ohms are connected across a battery which supplies a total of 12 amp. to them. How much current flows through each resistance?

Solution: Find the total resistance of 4, 8, and 10 ohms in parallel, as explained in Sec. 33. This total resistance is 2.105 ohms.

The total current through the group of resistance is 12 amp. and the total resistance of the group is 2.105 ohms. By substituting these values in Ohm's law, we find the e.m.f. across the group, as follows:

$$E = 12(2.105)$$
$$E = 25.26 \text{ volts}$$

Let x, y, and z be the respective currents through the 4-, 8-, and 10-ohm resistance. Then, since 25.26 is the voltage across the group, we have:

(a) $25.26 = 4x$	(b) $25.26 = 8y$	(c) $25.26 = 10z$
$4x = 25.26$	$8y = 25.26$	$10z = 25.26$
$x = 6.315$ amp.	$y = 3.158$ amp.	$z = 2.526$ amp.

Check: $6.315 + 3.158 + 2.526 = 11.999$ amp. total current, which checks with the 12 amp. given in the original problem.

42. Three resistances of 5, 6, and 7 ohms are connected in parallel. If the total current flowing is 10.7 amp., how much current is flowing through each resistance?

43. Three resistances of 5, 11, and 15 ohms are connected in parallel. If the total current supplied to this group of resistances is 17.7 amp., how much current is flowing through each resistance?

44. Three resistances of 8, 12, and 16 ohms are connected in parallel. If the total current flowing is 19.5 amp., how much current is flowing through each resistance?

45. Three resistances of 3, 4, and 5 ohms are connected in parallel. If the total current flowing is 9.4 amp., how much current is flowing through each resistance?

46. Three resistances of 5, 7, and 8 ohms are connected in parallel. If the total current flowing is 26.2 amp., how much current is flowing through each resistance?

47. Four resistances of 2, 3, 4, and 5 ohms are connected in parallel. If the total current flowing is 30.8 amp., how much current is flowing through each resistance?

48. Find the resistance of the following combination:

6 ohms in series with 8, 10, and 12 ohms connected in parallel

49. Find the resistance of each of the following combinations:

(a) 8 ohms in series with 6, 10, and 12 ohms connected in parallel
(b) 10 ohms in series with 8, 10, and 12 ohms connected in parallel
(c) 12 ohms in series with 6, 8, and 10 ohms connected in parallel

Example 10. Find the resistance of 5 and 8 ohms in parallel connected in series with 6 and 4 ohms in parallel.

Solution: In the diagram of connections below, note that there are two parallel resistance groups in series. We therefore find the resistance of each group and add these values to find the total resistance of the combinations.

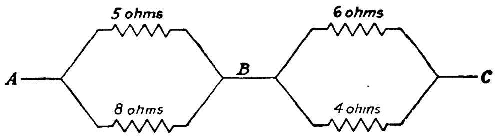


FIG. 17.

Using the formula of Sec. 33, we find that the resistance of the first group is 3.08 ohms. This is the resistance between A and B in Fig. 17.

Similarly, the resistance from B to C is found to be 2.4 ohms. The total resistance R from A to C is

$$R = 3.08 + 2.4 = 5.48 \text{ ohms } Ans.$$

50. Find the resistance of each of the following groups:

(*a*) 6 and 7 ohms in parallel connected in series with 8 and 9 ohms in parallel

(*b*) 6 and 8 ohms in parallel connected in series with 7 and 9 ohms in parallel

(*c*) 6 and 9 ohms in parallel connected in series with 7 and 8 ohms in parallel

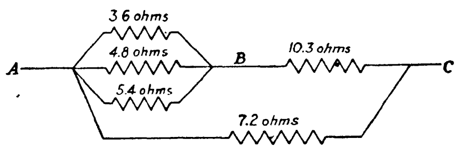


FIG. 18.

Example 11. Find the total resistance from A to C in the circuit whose diagram is given above:

Solution: First find the resistance of the group between A and B; then add to that value the resistance value between B and C. This sum is the total resistance of the path ABC.

The resistance from A to B, using the formula of Sec. 33, is found to be 1.49 ohms.

Adding to this value the resistance from B to C gives

Resistance A to C of the path $ABC = 1.49 + 10.3 = 11.79$ ohms

The 7.2-ohm resistance is, however, also connected across the points A and C and, therefore, must be treated as in parallel with the 11.79 ohms. The total resistance from A to C is, therefore, found by again using the formula of Sec. 33

$$\frac{1}{R} = \frac{1}{11.79} + \frac{1}{7.2}$$

$$\frac{11.79R}{R} = \frac{11.79R}{11.79} + \frac{11.79R}{7.2}$$

$$11.79 = R + 1.638R$$

$$11.79 = 2.638R$$

$$2.638R = 11.79$$

$$R = 4.47 \text{ ohms } Ans.$$

51. In the following figure, find the total resistance if $a = 6$ ohms, $b = 9$ ohms, $c = 17$ ohms, $d = 5$ ohms, and $e = 11$ ohms.

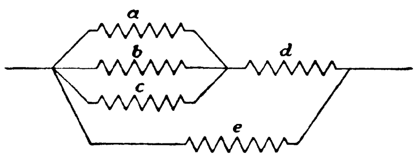


FIG. 19.

52. In the figure for Prob. 51, suppose $a = 12$ ohms, $b = 25$ ohms, $c = 19$ ohms, $d = 8$ ohms, and $e = 12$ ohms. Find the total resistance

53. Using the figure for Prob. 51, suppose $a = 25$ ohms, $b = 29$ ohms, $c = 36$ ohms, $d = 15$ ohms, and $e = 7$ ohms. Find the total resistance

54. Use the figure for Prob. 51 but disconnect resistance C and connect it in parallel with resistance d. Find the resistance of this new combination using the same values as in Prob. 51.

55. Solve Prob. 54 using the resistance values given in Prob. 52.

56. Solve Prob. 54 using the resistance values given in Prob. 53.

57. Find the resistance between points A and B in the following circuit.

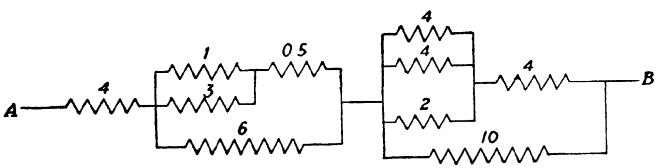


FIG. 20.

58. Find the resistance between points A and B in the following circuit.

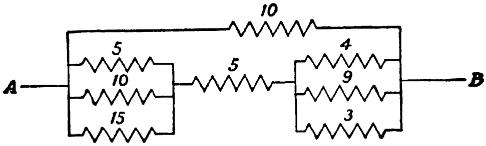


Fig. 21.

59. Find resistance between points A and B in the following circuit.

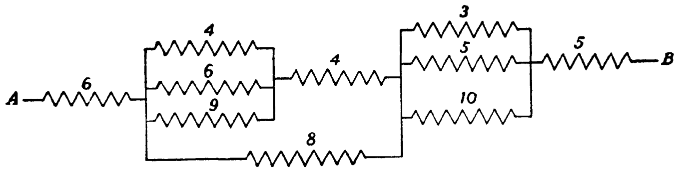


Fig. 22.

60. Five radio tubes having their heater filaments connected in parallel are to be operated from a storage battery whose terminal e.m.f. is 6 volts when supplying current to the tubes. Each heater filament requires 1.75 amp. at 2.5 volts. What size resistance should be connected in series with the battery and the tubes?

61. A class B amplifier has one type 56 tube and three type 59 tubes whose heater filaments are connected in parallel. The heater resistance of the 56-type tube is 2.5 ohms. The tubes of this amplifier draw a total of 7 amp. from the 2.5-volt filament circuit. What is the total filament-circuit resistance? What is the filament resistance of each 59-type tube?

62. Two 48-type tubes and two 37-type tubes have their filaments connected in series with resistor A across a 115-volt d.-c. line. A second resistor, B, is connected directly across the filaments of the type 37 tubes. The heater filament of the type 37 tube is designed to operate on 6.3 volts and 0.3 amp., that of the type 48 tube on 30 volts and 0.4 amp. What should be the value of resistors A and B to insure satisfactory voltage and current conditions for the heaters of these tubes?

CHAPTER VII

POWER IN THE ELECTRICAL CIRCUIT

36. Power.—The amount of energy or power consumed by an electrical device or generated by a dynamo is measured in watts, kilowatts, or horsepower. The relation among these units is as follows:

1 kilowatt (kw.) = 1,000 watts
1 horsepower (hp.) = 746 watts

37. The Watt.—The amount of energy or power expended when one volt causes a current of one ampere to flow in a circuit is called one "watt" of power. This relation is expressed by the formula

$$W = EI$$

where W = power in watts
E = pressure in volts
I = current in amperes

Another form of the power equation is obtained as follows:

$$W = EI$$

Since $E = IR$, substitute IR for E and we have

$$W = (IR)I$$
$$\therefore W = I^2R$$

38. Watt-hours. Kilowatt-hours.—The amount of electrical energy delivered to a consumer is measured by means of a meter commonly called a "watt-hour meter" or a "kilowatt-hour meter." This meter registers not only the amount of energy delivered to the consumer in watts or kilowatts but also combines with it the time factor, which, of course, must be considered when charging for power used.

One watt-hour is absorbed when a consumer uses one watt for a period of one hour, or when he uses one-half watt for a period of two hours. It is, then, the product of the number of watts consumed by the time in hours during which this power is used. Similarly, kilowatt-hours are the product of the number of kilowatts by the number of hours.

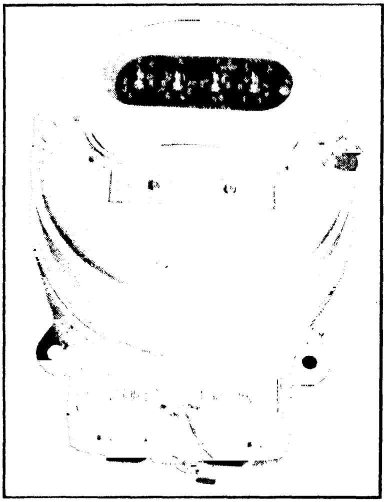


FIG. 23.—Westinghouse kilowatt-hour meter.

39. Candle Power.—The amount of light given off by an electric bulb is measured or rated in candle power. Quite often, the amount of power required for a lamp is specified in watts per candle power, that is, the number of watts necessary to produce one candle power of illumination.

Example 1. How many horsepower do 25 lamps consume if each takes 0.35 amp. at 110 volts?

Solution: The total current used by the group of lamps is

$$0.35(25) = 8.75 \text{ amp.}$$

Substitution in the formula $W = EI$ gives

$$\begin{aligned} W &= 110(8.75) \\ &= 962.5 \text{ watts} \end{aligned}$$

$$962.5 \text{ watts} = \frac{962.5}{746} \text{ hp.} = 1.29 \text{ hp. } \textit{Ans.}$$

Example 2. What is the resistance of a 40-watt, 110-volt lamp?

Solution: Find the current which the lamp draws from the line.

$$
\begin{aligned}
W &= EI \\
40 &= 110I \\
110I &= 40 \\
I &= 0.3636 \text{ amp.}
\end{aligned}
$$

By substituting in Ohm's law, we can now determine the resistance of the lamp:

$$
\begin{aligned}
E &= IR \\
110 &= 0.3636R \\
0.3636R &= 110 \\
R &= 302.5 \text{ ohms } \textit{Ans.}
\end{aligned}
$$

Example 3. A vacuum cleaner which requires 125 watts when in operation is used an average of $1\frac{1}{4}$ hr. a day. If electricity costs 7 cts. per kilowatt-hour, how much will it cost to use this cleaner for 1 month of 30 days?

Solution: The watt-hours per day $= 125(1\frac{1}{4}) = 156.25$ watt-hr.

Total watt-hours (30 days) $= 156.25(30) = 4{,}687.5$ watt-hr.

Total cost $= \$0.07(4.6875) = \$0.33$ *Ans.*

Problems

1. Express 2,600 watts in kilowatts and horsepower.

2. Express 3,525 watts in kilowatts and horsepower.

3. Express 425 watts in kilowatts and horsepower.

4. Express 250 watts in kilowatts and horsepower.

5. Express 37,500 watts in kilowatts and horsepower.

6. What horsepower is required to drive a generator which must deliver 4 amp. at 120 volts?

7. What horsepower do 10 lamps consume if each takes 0.25 amp. at 112 volts?

8. A steam engine is rated at 250 horsepower. What would be its rating in kilowatts?

9. An electric iron draws 5 amp. from a 110-volt line. How much power does it consume?

10. A 60-watt lamp is burning on a 112-volt line. How much current does it draw?

11. A 100-watt lamp is drawing 0.5 amp. from the line. What is the voltage across the line?

12. A motor is drawing the equivalent of 5.4 horsepower from a 112-volt line. What is the line current?

13. A motor draws 10 amp. from a 220-volt line. How many horsepower does it consume?

14. A motor draws the equivalent of 10 horsepower from a line in which the current is 15 amp. What is the line voltage?

15. A motor draws 7.5 amp. from a 220-volt line. How much power does it use?

16. How much power is used by a 25-ohm resistance through which a current of 7 amp. is flowing?

HINT.—Solve by substituting in the formula $W = I^2R$.

17. A current of 0.03 amp. is flowing through a 2,000-ohm resistance. What power is consumed?

18. A current of 0.7 amp. flowing through a 150-ohm resistance uses how many watts?

19. A voltmeter placed across a 125-ohm resistance registers 375 volts. How much power is used?

20. A resistance has 10 amp. flowing through it and consumes 750 watts. What is the value of the resistance?

21. A 60-watt lamp requires 1.8 watts per candle power. What is the candle power of the lamp?

22. A 2,000-ohm resistor can carry a maximum of 50 milliamp. without overheating. What should be its rating in watts?

23. How much current will ten 16-cp. lamps draw from a 110-volt line if each lamp requires 1.8 watts per candle power?

24. What is the resistance of a 60-watt, 110-volt lamp?

25. What is the resistance of a 100-watt, 110-volt lamp?

26. What is the resistance of a 25-watt, 110-volt lamp?

27. What is the resistance of a 60-watt, 220-volt lamp?

28. A 60-watt lamp burns for 24 hr. What is the cost of burning the lamp at 6 cts. per kilowatt-hour?

29. A 110-volt motor draws 6.4 amp. from the line. If it runs for 10 hr., how much would you bill the owner at $2\frac{1}{2}$ cts. per kilowatt-hour?

30. How long could you burn four 60-watt lamps for $1 if electricity costs 6 cts. per kilowatt-hour?

31. An electric toaster consuming 150 watts is used for $1\frac{1}{2}$ hr. each day. At 6 cts. per kilowatt-hour, how much will it cost to use the toaster for 30 days?

32. A radio set plate-voltage divider has a rating of 10 watts and can safely carry 20 milliamp. What is the resistance of this voltage divider?

33. A 10,000-ohm resistor can safely carry 10 milliamp. What should be the wattage rating of this resistor? What is the maximum safe voltage drop for this unit?

34. A resistance unit of 150 ohms can safely carry approximately 410 milliamp. What is the wattage rating of this resistor expressed to the nearest whole number?

CHAPTER VIII

SUBTRACTION OF SIGNED NUMBERS AND REMOVAL OF PARENTHESES

40. Subtraction of Signed Numbers.—Subtraction is the opposite of addition, and it can be readily shown that subtracting a positive number gives the same result as adding an equal negative number and that subtracting a negative number gives the same result as adding an equal positive number. The rule for subtraction is, therefore, as follows:

Rule.—To subtract one quantity from another, change the sign of the subtrahend and proceed as in addition.

This rule should be memorized and carefully observed when subtracting.

The process of changing the sign of the subtrahend may be done mentally or as illustrated below.

Example 1. From $12xy$ subtract $-13xy$.

Solution: Change the sign of the subtrahend and add.

$$\begin{array}{r} 12xy \\ \underline{\oplus 13xy} \\ 25xy \end{array} \; Ans$$

The fact that the $-13xy$ has been changed to a $+13xy$ is indicated by placing a circle around the plus sign.

Problems

In the following problems, subtract the lower from the upper number:

1. $\begin{array}{r} +10 \\ \underline{\oplus\ 6} \end{array}$ **3.** $\begin{array}{r} +14 \\ \underline{+\ 4} \end{array}$ **5.** $\begin{array}{r} -\ 8 \\ \underline{-10} \end{array}$

2. $\begin{array}{r} -\ 6 \\ \underline{+\ 5} \end{array}$ **4.** $\begin{array}{r} -\ 7 \\ \underline{-\ 5} \end{array}$ **6.** $\begin{array}{r} 10 \\ \underline{-15} \end{array}$

7. $\begin{array}{r} -6 \\ 13 \\ \hline \end{array}$ 10. $\begin{array}{r} 7 \\ -3 \\ \hline \end{array}$ 13. $\begin{array}{r} 5x \\ 12x \\ \hline \end{array}$

8. $\begin{array}{r} 14 \\ 25 \\ \hline \end{array}$ 11. $\begin{array}{r} -10c \\ -15c \\ \hline \end{array}$ 14. $\begin{array}{r} -5y \\ -12y \\ \hline \end{array}$

9. $\begin{array}{r} -8 \\ -4 \\ \hline \end{array}$ 12. $\begin{array}{r} 14x \\ -10x \\ \hline \end{array}$ 15. $\begin{array}{r} -3y \\ -8y \end{array}$

Find the value of the following:

16. $7 - (-7)$
17. $-7 - (-10)$
18. $-6 - (-10)$
19. $6 - (+10)$
20. $-10 - (-17)$
21. $12 - (-4)$
22. $-5 - (+9)$
23. $8 - (+2)$

Add the following:

24. $\begin{array}{r} +10 \\ -6 \\ \hline \end{array}$ 25. $\begin{array}{r} +8 \\ -8 \\ \hline \end{array}$ 26. $\begin{array}{r} -25 \\ +37 \\ \hline \end{array}$ 27. $\begin{array}{r} -19 \\ -22 \\ \hline \end{array}$

Perform the operations indicated in the following:

28. $64 - (-7)$
29. $-10 + (+13)$
30. $-17 - (-37)$
31. $-7 + (-21)$
32. $12 - (-26)$
33. $85 - (-85)$
34. $-85 + (+85)$
35. $85 + (+85)$
36. $0 - (-10)$
37. $0 - (+10)$
38. From $6x^2 - 2xy + 7$ Subtract $4x^2 - 2xy - 6$
39. From $5x^2 - 2x + 8$ Subtract $3x^2 + 2x - 6$
40. From $9x^2 + 4x^2y - 18y$ Subtract $-7x^2 - 4x^2y + 18y$
41. From $5x - 3y + 4z$ Subtract $18x - 17y + 16z$
42. From $9x^2 + 4x - 10$ Subtract $-8x^2 + 3x + 11$
43. From $8x^2 - 9x + 4y$ Subtract $-5x^2 + 10y - 16$

41. Parentheses.—Parentheses are used to group together certain quantities all of which are to be affected by the same operation. For example, we can indicate that one polynomial is to be added to or subtracted from another by using parentheses:

$$(3x^2 - 4x + 4) - (7x^2 + 3x - 9)$$

This means that the second expression is to be subtracted from the first.

A parenthesis may be removed from an expression according to the following rules:

Rule 1.—*A parenthesis preceded by a plus sign may be removed without making any other change.*

Rule 2.—*If a parenthesis is preceded by a minus sign, it can be removed if the sign of every term included within the parenthesis is changed.*

Rule 3.—*If a parenthesis occurs within a parenthesis, remove the inner parenthesis first.*

Example 2. Simplify $9x^2 - 17xy + (18xy + 9) + 6x^2 - (10x^2 - 12xy + 17)$.

Solution: Remove the two parentheses according to rules 1 and 2; then combine the similar terms.

$$\begin{aligned} &9x^2 - 17xy + (18xy + 9) + 6x^2 - (10x^2 - 12xy + 17) \\ &= 9x^2 - 17xy + 18xy + 9 + 6x^2 - 10x^2 + 12xy - 17 \\ &= 5x^2 + 13xy - 8 \textit{ Ans.} \end{aligned}$$

Example 3. Simplify $6a^2 - [12b^2 - (8c + 7a^2 - 13b^2) + 9c] - 5b^2$.

Solution: Remove the inner parenthesis and proceed as in Ex. 2.

$$\begin{aligned} &6a^2 - [12b^2 - (8c + 7a^2 - 13b^2) + 9c] - 5b^2 \\ &= 6a^2 - [12b^2 - 8c - 7a^2 + 13b^2 + 9c] - 5b^2 \\ &= 6a^2 - 12b^2 + 8c + 7a^2 - 13b^2 - 9c - 5b^2 \\ &= 13a^2 - 30b^2 - c \textit{ Ans.} \end{aligned}$$

Problems

Simplify the following by removing parentheses and combining similar terms:

1. $(4x^4 - 2x^3 - 6x + 1) - (x^3 + x^2 - 3)$

2. $(3x^2 - 7xy - 4y^2) - (-8x^2 + 2xy - 7y^2)$

3. $3y^2 - (4x^2 - 3y^2) + 2x - (3x^2 + 4xy - 3y^2)$

4. $7y^2 + (3x^2 - 7y^2) - (3x^2 + 4y^2 - 7)$

5. $4y - [5x^2 - (3x^2 + 7y^2 - 5xy) + 5y]$

6. $x^3 - 2x^2 + 4 - (7x^2 - 4x + 1) - (5x^3 - 2x^2 + 4x - 7)$

7. $9x - (7x^3 + 4x^2 - 3x) - (8x^3 - 3 - 7x^2)$

8. $8x + (5x^2 - 7y^2) - 4xy - (8x^2 - 7xy + 3y^2 + 8x)$

9. $5x^2 - (4x^2 - 7xy + 10) + 4y^2 - (-3x^2 - 7xy + 4y^2)$

10. $8x^2 + 6x^2 - (7x^2 + 4xy - 7y^2) - 2y^2 - 4xy$

11. $9a + 12b - (6a + 5b - 8c) - 3c + 12b - (4c + 5b)$

12. $17m^2 - (19m + 12n - 5n^2) - (-12n + 7m + 12n^2)$

13. $12y - 18z + 15w - (8w + 7z) + (5y - 15w - 14z)$
14. $16x^2 + (19x - 12y) - (15x^2 - 18y^2 - 22y + 19x) - 10y$
15. $39a^2 - (42c^2 - 21a^2 - 12b^2) + 13b^2 - 36a^2 + 16c^2$
16. $5x - [7x + (5y - 6z) - (4x + 10y - 9z)] - 8y + 5z$
17. $7x^2 + [18y^2 - (7z + 7x^2 - 15y^2) + 8z] - 16y^2$
18. $12x^3 - [15x^2 - (7x^3 + 5x) + (9x^2 + 12x) - 18x^3] - 7x^2$
19. $19a - [12a + (5b - 6c) - (18c - 11b - 13a) + 9b]$
20. $32a^2 - [- 17ab - (9a^2 + 6ab - 12b^2) + 7b^2] + 8a^2 - 19b^2$

21. Add $9x^2 - 17xy + 8y^2$ and $15x^2 + 5xy - 3xy^2 - 3y^2$. Then subtract $18x^2 + 3xy - 7xy^2 - 5y^2$ from this sum.

22. Subtract $6a^2 - 12b^2 - 8c$ from $7a^2 + 5b^2 + 12d$. Then add $6c - 5d - 13a^2 + b^2$ to this difference.

23. From the sum of $19a - 27b - 36d$ and $-12a - 5b + 14d$ subtract the sum of $8a + 13b - 8d$ and $-4a - 5b + 12c$.

24. Add $0.4x - 3.2y + 0.7z$ and $0.9x + 0.5y - 3.6z$. From this sum subtract $x - y - z$.

25. Subtract $9.5a - 6.7b - 0.9c + 5.6d$ from $- a + b - 8.2d$. Then add $1.2a - 3.1b + 0.5c - d$ to this difference.

26. From the sum of $1.5a - 4.3c + 8.9g - h$ and $-2.4a - 3.6b - 8.9g + h$ subtract the sum of $3.9a - 2.5b + c$ and $3.5b - 4.9c$.

27. From $1.2x - 4.2y + 5.8z$ subtract $-2.5x + y - w$. Add to this difference $x + 4.7y - 3.7z + 1.5w$ and $3.2x - 0.6y + 0.3z - 0.8w$.

28. Add $3x^2 + 9y^2 + 4xy - 8yz$, $0.1x^2 - 3.2y^2 - 0.4xy + 5.2yz$, $2.3x^2 - 4.1y^2 - 2.6xy + yz$. From this sum subtract $x^2 - 2y^2 - 3xy + 4yz$.

29. From $3.2x - 4.1y - 3.2z$ subtract $-3.2x - 2.7y + z$. Add to this difference $x + 3y + 4z$ and $0.2x - 3.1y + 2.7z$.

30. From $8.6a - 2.5b + 8.1c - 3.4d$ subtract $-2.4a - b + 2.1c - 4.2d$. From this difference subtract the sum of $a - 0.5b + 2.9c - d$ and $-2.4a - b - 1.4c + 2.6d$.

CHAPTER IX

MULTIPLICATION AND DIVISION OF SIGNED NUMBERS

42. Multiplication.—Multiplication is a short process of adding the same number a certain number of times. We shall now see how to multiply signed numbers.

$$(+4)\,(+3) = (+4) + (+4) + (+4) = +12$$
$$(-4)\,(+3) = (-4) + (-4) + (-4) = -12$$

In the first case we have three $+4$s to be added giving $+12$, and in the second case we have three -4s to be added giving -12. Now, $(-4)\,(+3)$ evidently is the same as $(+3)\,(-4)$, since we could not get two different results when multiplying the same two quantities. Therefore, the following must be true: $(+3)\,(-4) = (-3) + (-3) + (-3) + (-3) = -12$

Multiplication by a negative number means, therefore, to add a certain number with its sign changed a given number of times. Therefore, $(-4)\,(-3) = (+4) + (+4) + (+4) = 12$, using the same reasoning as for the preceding case.

We can now state the rules for multiplying signed numbers.

Rule 1.—*When multiplying together two numbers of like signs, the product is positive.*

Rule 2.—*When multiplying together two numbers of opposite signs, the product is negative.*

According to the first of these rules, the product of $+2$ and $+3$ is $+6$, and the product of -2 and -3 is $+6$.

According to the second rule, the product of $+4$ and -5 is -20, and the product of -4 and $+5$ is -20.

Multiply the following:

1. $(-6)(\ 4)$	**5.** $(-6)(-10)$	**9.** $(\ 6)(-8)$
2. $(-5)(-7)$	**6.** $(-8)(-16)$	**10.** $(-10)(-5)$
3. $(\ 4)(\ 5)$	**7.** $(\ 2)(\ 5)$	**11.** $(-6)(-4)$
4. $(\ 6)(-2)$	**8.** $(\ 7)(-2)$	**12.** $(\ 7)(-3)$

43. Multiplication of Monomials.—The product of two monomials is obtained by multiplying their numerical coefficients and adding the exponents of the same letters.

$$(x^4)(x^3) = (x \cdot x \cdot x \cdot x)(x \cdot x \cdot x) = 7xs \text{ multiplied} = x^7$$
$$(4x^2)\ (3x^3) = (4)(3)(x^2)(x^3) = 12x^5$$

Problems

Multiply as indicated:

1. $(x^3)(x^5)$
2. $(-x^4)(-x^7)$
3. $3x(4x^2)$
4. $-4x^5(5x^4)$
5. $-3y^4(-4y^2)$
6. $4a^5(-3a^3)$
7. $5xy(-3x^2y^2)$
8. $-9x^2y^3(-5x^3y^2)$
9. $-6x^2y(4xy^2)(-3x^3y)$
10. $6a(-4ax)(-5a^2x^2)$
11. $6xyz(-10x^3yz^3)$
12. $7x^4y(-8xy^4)$
13. $(-3)(-3)(-3)$
14. $(-3x)^3$
15. $(-2x)^4$
16. $(4y^3)^3$
17. $(-6y^2)^2$
18. $(3x^2y^3)^3$

44. Multiplication of a Polynomial by a Monomial.—The product of a polynomial and a monomial is obtained by multiplying each term of the polynomial by the monomial.

Example 1. Multiply $7a^2 - 5ab$ by $4c$.

Solution: $(7a^2 - 5ab)4c = 28a^2c - 20abc$

Multiplying $7a^2$ by $4c$ gives $28a^2c$, and multiplying $-5ab$ by $4c$ gives $-20abc$.

Example 2. Multiply as indicated $5x^2y(8x^2 - 3xy - 4y^2)$.

Solution: $5x^2y(8x^2 - 3xy - 4y^2) = 40x^4y - 15x^3y^2 - 20x^2y^3$

Check: This multiplication may be checked by letting $x = 2$ and $y = 3$, since x and y may represent any number.

$$8x^2 - 3xy - 4y^2 = 8(2)^2 - 3(2)3 - 4(3)^2 = 32 - 18 - 36 = -22$$
$$5x^2y = 5(2)^23 = 60$$

The product of 60 and -22 is $-1{,}320$. Therefore, the product of $8x^2 - 3xy - 4y^2$ and $5x^2y$, which we figured to be $40x^4y - 15x^3y^2 - 20x^2y^3$, should equal $-1{,}320$ when these values of x and y are substituted.

$$\begin{aligned} 40x^4y - 15x^3y^2 - 20x^2y^3 &= 40(2)^43 - 15(2)^3(3)^2 - 20(2)^2(3)^3 \\ &= 40(16)3 - 15(8)9 - 20(4)27 \\ &= 1{,}920 - 1{,}080 - 2{,}160 \\ &= -1{,}320\ \textit{Ans.} \end{aligned}$$

Since the two results agree, the answer is correct.

Problems

1. $(4a - 7b)3$
2. $-5(3x - 5y)$
3. $(8x^2 - 12y^2)4$
4. $-3(3x - 5y)$

5. $-2(5ab - 7cd)$
6. $(8a^2b - 3bc)5$
7. $4ab(3a - 4b)$
8. $-6x^2y(5x^2 - 3x + 4)$
9. $2x(3a + 4b - 6c)$
10. $-3x^3(4x^2 - 6xy + 4y^2)$
11. $4x^3y(3ax + 4ay)$
12. $5a(6a^3b - 7b^2c + 9a^2c)$
13. $6x(3xy - 8xy^2 + 4x^2y^3)$
14. $-3x^3y^2z(4x^2 - 2y - 3z)$
15. $-14x^2(5x^3 - 6x^2y - 4xy^2 + 4y^3)$
16. $7x^3(-2x^4 - 7y^2 + 3z)$
17. $9y^2(5y^3 - 6xy^2 - 5x^2y - 4x^3)$
18. $-5x^4(6x - 7y + 14ab)$

Example 3. Simplify $-3xy(x^2 - y^2) + 5y(x^3 + x^2y)$.
Solution: First multiply as indicated; then combine similar terms.

$$\begin{aligned} -3xy(x^2 - y^2) + 5y(x^3 + x^2y) &= -3x^3y + 3xy^3 + 5x^3y + 5x^2y^2 \\ &= 2x^3y + 5x^2y^2 + 3xy^3 \text{ } Ans. \end{aligned}$$

Problems

1. $5a(a^2 + b^2) + 6b(a + b)$
2. $6x^2(3x - xy) - 7x(4x^2 + x^2y)$
3. $-2x(x^3 - y^3) + 5y^2(x^2 + y^2)$
4. $4x^2y(x^3 + y^2) - 3xy^2(xy - xy^2)$
5. $5a^2b^3(2ab - 3b^2) - 4b^2(3a^3b^2 + 4a^2b^3)$
6. $10d(3c^2d + 5d^3) + 6cd^2(7c - 10d)$
7. $7r^2t(6s^2t - 9rs^2) - 3st^2(5rs - 8r^2s)$
8. $8xy^2(6x^2y - 5x + 7y^2) - 4y(3x^3y^2 + 7x^2y)$

45. Division of Signed Numbers.—From our knowledge of arithmetic we know that division is just the opposite of multiplication. Using this knowledge we can arrive at the laws for the division of signed numbers.

$$\begin{aligned} (+3)(+4) &= +12; \text{ therefore, } & 12 \div (+4) &= +3 \\ (-3)(+4) &= -12; \text{ therefore, } & -12 \div (+4) &= -3 \\ (-3)(-4) &= +12; \text{ therefore, } & +12 \div (-4) &= -3 \\ (+3)(-4) &= -12; \text{ therefore, } & -12 \div (-4) &= +3 \end{aligned}$$

From the above analysis we see that the law of signs for division is the same as for multiplication and may be stated as follows:

Rule 1.—*When dividing two numbers of like signs, the quotient is positive, and when dividing two numbers of unlike signs, the quotient is negative.*

Consider, also, the following analysis:

$$(+4xy)(+5xy) = 20x^2y^2; \text{ therefore, } 20x^2y^2 \div (5xy) = 4xy$$
$$(-12xy)(-7xy) = 84x^2y^2; \text{ therefore, } 84x^2y^2 \div (-12xy) = -7xy$$

Rule 2.—When dividing algebraic terms, divide the coefficient of the dividend by the coefficient of the divisor and subtract the exponents of the letters in the divisor from the exponents of the same letters in the dividend.

Problems

Using the rules for division, solve the following problems:

1. Divide $12x^4$ by $3x^2$

2. Divide $-9x^2y^2$ by $3xy$

3. Divide $-14x^3y$ by $2xy$

4. $12x^2y^2 \div 3x = ?$

5. $25x^5y^3 \div -5x^2y^2 = ?$

6. $-36x^4y^2 \div 12xy^2 = ?$

7. $-42a^4b \div -7a^4 = ?$

8. $35ab^2 \div -5b^2 = ?$

9. $\dfrac{125x^3y^5}{25xy^3} = ?$

10. $\dfrac{39x^2y^4}{13x^2y^2} = ?$

11. $\dfrac{225x^3y^4z}{15xy^2z} = ?$

12. $\dfrac{27x^3y^2}{3xy} = ?$

13. $\dfrac{58a^2b^4}{29ab^4} = ?$

Divide as indicated:

14. $3x^2y\,\overline{)9x^3y^2 - 27x^2y + 12x^4y^3}$

15. $2xy^2\,\overline{)8x^4y^5 - 14x^3y^4 - 20x^2y^3}$

16. $3x\,\overline{)12x^5y + 15x^2y^3 - 21xy^4}$

17. $2cb\,\overline{)8a^2b - 10a^3b^2 - 14a^5b^4}$

18. $\dfrac{125a^3b^3 + 25ab^4 - 5a^2b^5}{-5ab}$

Example 4. Solve and check the equation

$$8(3x - 7) - 13x = 3(2x - 7)$$

Solution: Multiply as indicated; then solve the resulting equation in the usual manner.

$$\begin{aligned} 8(3x - 7) - 13x &= 3(2x - 7) \\ 24x - 56 - 13x &= 6x - 21 \\ 24x - 13x - 6x &= -21 + 56 \\ 5x &= 35 \\ x &= 7 \textit{ Ans.} \end{aligned}$$

Check:

$$\begin{aligned} 8(3x - 7) - 13x &= 3(2x - 7) \\ 8(3\cdot7 - 7) - 13(7) &= 3(2\cdot7 - 7) \\ 8(21 - 7) - 91 &= 3(14 - 7) \\ 8(14) - 91 &= 3(7) \\ 112 - 91 &= 21 \\ 21 &= 21 \end{aligned}$$

Example 5. Solve and check the equation

$$\tfrac{3}{5}(4x - 16) + \tfrac{5}{8}(3x + 5) = 5x - 13$$

Solution:
$$\tfrac{3}{5}(4x - 16) + \tfrac{5}{8}(3x + 5) = 5x - 13$$
$$\frac{12x}{5} - \frac{48}{5} + \frac{15x}{8} + \frac{25}{8} = 5x - 13$$

Clearing this equation of fractions gives

$$\begin{aligned} 96x - 384 + 75x + 125 &= 200x - 520 \\ 96x + 75x - 200x &= -520 + 384 - 125 \\ -29x &= -261 \\ x &= 9 \end{aligned}$$

Check:

$$\begin{aligned} \tfrac{3}{5}(4x - 16) + \tfrac{5}{8}(3x + 5) &= 5x - 13 \\ \tfrac{3}{5}(4\cdot 9 - 16) + \tfrac{5}{8}(3\cdot 9 + 5) &= 5(9) - 13 \\ \tfrac{3}{5}(36 - 16) + \tfrac{5}{8}(27 + 5) &= 45 - 13 \\ \tfrac{3}{5}(20) + \tfrac{5}{8}(32) &= 32 \\ 12 + 20 &= 32 \\ 32 &= 32 \end{aligned}$$

Solve and check the following equations:

1. $4(x - 6) = 3x - 10$
2. $7(2x + 5) = 3(x - 3)$
3. $4(2x - 1) - 3x = 7(2x + 8)$
4. $4x(2x + 5) = 8x^2 - 14x - 34$
5. $2x + \tfrac{3}{4}(x + 5) = \tfrac{5}{2}(c + 20) + 5$
6. $13\left(\frac{2x}{11} - 1\right) = 12\left(\frac{x}{11} + 4\right) + 9$
7. $3(x - 2) + 15 = 5x - 3$
8. $11 - 3(x - 2) = x - 8$
9. $8(3 - 2x) = 2(4 - x) - 30$
10. $5(10x + 7) + 5 = 6(5x + 8)$
11. $\tfrac{4}{5}(5x - 90) = \tfrac{2}{3}(x + 2)$
12. $12 = \tfrac{6}{7}(4x + 46)$
13. $15 = \tfrac{5}{12}(8x + 60)$
14. $-3 = \tfrac{1}{13}(2x - 9)$
15. $\tfrac{2}{7}(3x - 69) = \tfrac{3}{8}(7x - 29)$
16. $\tfrac{7}{8}(5x + 33) - \tfrac{3}{5}(2x - 5) = 7x + 51$
17. $\tfrac{4}{5}(7x - 19) - \tfrac{2}{3}(5x - 39) = 9x - 70$
18. $\tfrac{4}{7}(3x - 60) - \tfrac{3}{8}(5x - 71) = x - 39$
19. $\tfrac{3}{4}(5x + 33) + \tfrac{9}{11}(3x + 62) = 2x + 70$
20. $\tfrac{5}{8}(7x - 31) - \tfrac{4}{7}(9x - 39) = 6x - 58$

CHAPTER X

SQUARE ROOT. USE OF THE FORMULA

46. Square Root.—It is frequently necessary, in the course of a solution of a problem, to extract the square root of a number. This process, when understood, is a simple one, and every student should master it thoroughly. We shall briefly describe the order of operations used in finding the square root of a number and shall follow this with an example. To take the square root of a number, proceed as follows:

1. Divide the number into groups of two digits, beginning at the decimal point and working toward the left and, also, toward the right if there are figures on both sides of the decimal point. The extreme left-hand group may contain only one digit.
2. Find the nearest square root of the number comprising the left-hand group and use this root as the first figure of the required root. Square this root and subtract from the left-hand group.
3. Bring down the figures in the second group and form the trial divisor by multiplying by two the root already found.
4. Cover the last figure in the remainder and determine how many times the trial divisor is contained in it. Use this number as the second figure in the required root and, also, place this figure to the right of the trial divisor, thus forming a complete divisor of two or three digits.
5. Multiply the complete divisor by the second figure in the root and subtract from the previous remainder. If the product is too large, the second figure in the root must be reduced by one and the last figure in the complete divisor must also be reduced by one.
6. After subtracting, bring down the two figures in the next group and repeat the process, forming a new trial divisor by again multiplying by two the root already determined.

Example 1. Find the square root of 3,564.09.

$$\begin{array}{r l}
 & 35'64'.09)\overline{59.7}\ \textit{Ans.} \\
 & \underline{25} \\
109) & 1064 \\
 & \underline{981} \\
1187) & 8309 \\
 & \underline{8309}
\end{array}$$

Find the square root of the following numbers:

1. 130,321	**4.** 948.64	**7.** 4,841.3764
2. 434,281	**5.** 502,681	**8.** 6.702921
3. 82.9921	**6.** 65.2864	**9.** 0.03511876

Find the square root of the following to four decimal places:

10. 11	**13.** 0 4	**16.** 75
11. 2.5	**14.** 0.9	**17.** 31.8
12. 0.036	**15.** 5.07	**18.** 15

Solve and check the following equations:

1. $6a^2 = 864$	**7.** $5d^2 = 13{,}468.05$
2. $12x^2 = 2{,}028$	**8.** $3b^2 = 66.27$
3. $7x^2 = 567$	**9.** $11c^2 = 233.7731$
4. $9x^2 = 5{,}625$	**10.** $8y^2 = 17.5232$
5. $4x^2 = 1.3456$	**11.** $7k^2 = 2{,}884.63$
6. $2x^2 = 2{,}767.68$	**12.** $5x^2 = 3{,}018{,}645$

47. The Formula.—A formula is an equation which expresses a certain fact or relation by means of symbols instead of expressing the fact or relation in words. Thus, the relation that the voltage in a circuit is equal to the product of the current by the resistance has been expressed as

$$E = IR$$

Example 2. In the formula $A = \frac{h}{2}(a + b)$, $A = 143$, $h = 11$, $b = 19$, find the value of a.

Solution: Substitute the given values in the formula and solve for a.

$$\begin{aligned}
143 &= {}^{11}\!/_{2}(a + 19) \\
143 &= \frac{11a}{2} + \frac{209}{2} \\
286 &= 11a + 209 \text{ clearing of fractions} \\
-11a &= 209 - 286 \\
-11a &= -77 \\
a &= 7 \quad \textit{Ans.}
\end{aligned}$$

Check: Substitute the given values and the calculated value of a in the formula. This gives

$$143 = {}^{11}\!/_{2}(7 + 19)$$
$$143 = {}^{11}\!/_{2}(26)$$
$$143 = 11(13)$$
$$143 = 143$$

Example 3. In the formula $A = \frac{2ab}{a + b}$, $A = 8$, $a = 5$, find b.

Solution: Substituting the given values in the formula gives

$$8 = \frac{2(5)b}{5 + b}$$

$$8(5 + b) = 2(5)b \text{ by cross-multiplying}$$
$$40 + 8b = 10b$$
$$8b - 10b = -40$$
$$-2b = -40$$
$$b = 20$$

Check:

$$8 = \frac{2(5)20}{5 + 20}$$
$$8 = {}^{200}\!/_{25}$$
$$8 = 8$$

Problems

In the following formulas substitute the values given and solve for the remaining letter:

1. $A = hb$, $h = 17$, $b = 24$. Find A.
2. In formula 1, $A = 327$, $h = 3$. Find b.
3. $d = rt$, $d = 246$, $t = 10$. Find r.
4. $s = \frac{1}{2}at^2$, $s = 200$, $t = 5$. Find a.
5. In formula 4, $s = 648$, $a = 16$. Solve for t.
6. In formula 4, $s = 2{,}720.9$, $a = 32.2$. Find t.
7. In formula 4, $s = 1{,}507{,}539.6$, $a = 32.2$. Solve for t.
8. $s = \frac{n}{2}(a + l)$, $s = 60$, $a = 5$, $l = 7$. Find n.
9. $s = 375$, $l = 39$, $n = 15$. Find a, using formula for Prob. 8.
10. $s = 136$, $a = 3$, $n = 8$. Find l, using formula for Prob. 8.
11. $s = 402$, $a = 6$, $l = 61$. Find n, using formula for Prob. 8.
12. $C = 5/9(F - 32)$, $C = 69$. Find F.
13. In formula for Prob. 12, find F when C has the following values:

(a) $C = 0$ (b) $C = 100$ (c) $C = 10$ (d) $C = -20$ (e) $C = 325$
(f) $C = -50$ (g) $C = 4$ (h) $C = 12$

14. $s = \dfrac{rl - a}{r - 1}$. $S = 200$, $l = 140$, $a = 8$. Solve for r.

15. Using the same formula, find l when $s = 248$, $r = 2$, $a = 8$.
16. Using the same formula, find r when $s = 1{,}555$, $a = 1$, $l = 1{,}296$.
17. Using the same formula, find l when $s = 728$, $a = 2$, $r = 3$.
18. Using the same formula, find r when $s = 5{,}689$, $a = 13$, $l = 5{,}216$.

19. $R = \dfrac{kl}{d^2}$, $R = 0.05$, $l = 2{,}500$, $d = 250$. Find k.

20. Using the same formula, find d when $k = 10.4$, $l = 3{,}000$, $R = 2.5$.
21. Using the same formula, find l when $k = 17$, $d = 187.5$, $R = 4$.
22. Using the same formula, find l when $k = 10.4$, $d = 250$, $R = 6$.
23. Using the same formula, find d when $k = 17$, $l = 5{,}280$, $R = 8$.
24. Using the same formula, find d when $k = 10.4$, $l = 5{,}280$, $R = 6$.
25. Using the same formula, find l when $k = 10.4$, $d = 450$, $R = 5$.

48. Factors Which Determine the Resistance of Any Wire.—The resistance of any conductor varies directly as its length and inversely as its cross-sectional area. Expressed in different words, this means that the longer a wire of a given size the greater will be its resistance, and the greater the diameter of a given length of wire the smaller will be its resistance. Most conductors used for wiring purposes are round. Now, the area of a circle is proportional to the square of its diameter; therefore, if we know the relative diameters of two wires, we can determine how many times as large one is than the other without knowing the actual areas in square inches, square feet, or some other unit. This fact has lead to the adoption of a unit for expressing the area of a circular wire. This unit is called the "circular mil."

49. Circular Mil.—A mil is one one-thousandth of an inch (1/1,000 inch). A circular mil is the area of a wire one mil in diameter. The area of a wire two mils in diameter will be four times as great, and, therefore, a wire two mils in diameter contains four circular mils.

Rule.—To determine the circular-mil area of any wire, express the diameter of the wire in mils and square this number.

Example 4. How many circular mils in a wire of 0.3 in. diameter?

$$0.3 \text{ in.} = 300 \text{ mils.} \quad (300)^2 = 90{,}000$$

Therefore, a wire 0.3 in. in diameter contains 90,000 cir. mils.

50. Formula for the Resistance of Any Conductor.—A formula which makes use of the relation just discussed to determine the resistance of any conductor is the following.

$$R = \frac{kl}{d^2} \text{ or } \frac{kl}{CM}$$

where R is the resistance of the conductor in ohms

k is the resistance of one mil-foot of the conductor in ohms
l is the length in feet
d is the diameter of the conductor in mils, or d^2 is the area in circular mils

One mil-foot of wire is a wire one foot long and one mil in diameter. The constant k depends for its value upon the kind of material used to make up the conductor. The following table gives the approximate value of k for different materials:

Material	Value of k at 20° C.
Aluminum	17.0
Copper (annealed)	10.8
Iron (annealed)	60.0
German silver (18 per cent nickel)	200
German silver (30 per cent nickel)	290
Nickel	64.3
Steel wire	86.0

Problems

1. How many mils in 0.1 in.?
2. How many mils in 0.005 in.?
3. How many mils in 2.01 in.?
4. How many inches in 2,250 mils?
5. How many inches in 27 mils?
6. How many inches in 247 mils.?
7. What is the area in circular mils of a wire 0.6 in. in diameter?
8. What is the area in circular mils of a wire 0.027 in. in diameter?
9. What is the area in circular mils of a wire 0.724 in. in diameter?
10. What is the area in circular mils of a wire 1.2 in. in diameter?

Example 5. What is the resistance of a copper wire 1,450 ft. long if its diameter is 0.125 in.?

Solution Substitute in the formula $R = \dfrac{kl}{d^2}$

In this formula, d must be expressed in mils 0 125 in is equal to 125 mils

$$d = 125$$

From the table of specific resistance we find that for copper

$$k = 10\ 8$$
$$l = 1{,}450$$

Substituting these values in the formula gives

$$R = \frac{10\ 8(1\ 450)}{125(125)}$$
$$= \frac{10\ 8(58)}{125(5)} \text{ by cancelling 25 from numerator and denominator}$$
$$= \frac{626\ 4}{625}$$
$$= 1\ 002 \text{ ohm}$$
$$= 1 \text{ ohm } \textit{Ans}$$

Since the value of k is given to three significant figures, the fourth place in the result is dropped, since it is less than 5

Example 6 What is the diameter of a wire composed of nickel which has a resistance of 2 24 ohms? The wire is 1,200 ft long

Solution In this case, $l = 1{,}200$, $R = 2\ 24$, and $k = 64\ 3$ Substituting these values in the resistance formula gives

$$2\ 24 = \frac{64\ 3(1{,}200)}{d^2}$$

This may be written

$$\frac{2\ 24}{1} = \frac{64\ 3(1{,}200)}{d^2}$$

Cross-multiplying gives

$$2\ 24d^2 = 64\ 3(1{,}200)$$
$$d^2 = \frac{64\ 3(1{,}200)}{2\ 24}$$
$$d^2 = \frac{64\ 3(75)}{0\ 14} \text{ by cancelling 16 from numerator and denominator}$$
$$d^2 = \frac{4{,}822\ 5}{0\ 14}$$
$$d^2 = 34{,}446$$
$$d = \sqrt{34{,}446}$$
$$d = 185\ 6 \text{ mils}$$
$$d = 186 \text{ mils or } 0\ 186 \text{ in } \textit{Ans.}$$

Example 7. What will be the diameter of a copper wire 200 ft. long which has the same resistance as an aluminum wire 0.250 in. in diameter and 300 ft. long?

Solution: The resistance R of the copper wire is

$$R = \frac{10.8(200)}{d^2}$$

The resistance R of the aluminum wire is

$$R = \frac{17(300)}{250(250)}$$

Since the resistance of the aluminum wire equals that of the copper wire, these two expressions are equal to each other, and we may write

$$\frac{10.8(200)}{d^2} = \frac{17(300)}{250(250)}$$

$$17(300)d^2 = 10.8(200)(250)(250) \text{ by cross-multiplying}$$

$$d^2 = \frac{10.8(200)(250)(250)}{17(300)}$$

$$d^2 = \frac{3.6(2)(250)(250)}{17} \quad \text{by cancelling 300 from numerator and denominator}$$

$$d^2 = \frac{450,000}{17}$$

$$d^2 = 26,471$$

$$d = \sqrt{26,471}$$

$$d = 162.7$$

$$d = 163 \text{ mils or } 0.163 \text{ in. } \mathit{Ans.}$$

Problems

1. What is the diameter in mils of a wire whose circular-mil area is 22,500?

2. What is the diameter in mils of a wire whose circular-mil area is 516,961?

3. What is the diameter in inches of a wire whose circular-mil area is 23,804,641?

4. What is the diameter in mils of a wire whose circular-mil area is 0.3364?

5. A wire has an area of 0.120409 cir. mil. What is its diameter in inches?

6. A wire has an area of 1,159.4025 cir. mil. What is its diameter in mils?

7. A wire has an area of 3.8416 cir. mil. What is its diameter in inches?

8. What will be the resistance of 1,200 ft. of copper wire 0.065 in. in diameter?

9. 3,600 ft. of copper wire whose diameter is 1/8 in. will have how many ohms resistance?

10. One mile of 14 B & S wire has how much resistance (d = 64 mils)?

11. How long must a copper wire 1/4 in. in diameter be if its resistance measures 16 ohms?

12. What is the diameter of a copper wire whose resistance is 12 ohms? The wire is 11,250 ft. long.

13. What will be the resistance of 2,500 ft. of aluminum wire 1/2 in. in diameter?

14. What is the diameter of a copper wire 1,600 ft. long whose resistance is 0.4 ohm?

15. What is the resistance of 2 miles of aluminum wire 1.5 in. in diameter?

16. How many feet of copper wire 1/8 in. in diameter will have a resistance of 1 ohm?

17. The distance between a two-wire generator and a group of lamps is 750 ft. If the wires are 0.243 in. in diameter, what is the line resistance?

18. What would have to be the distance between the generator and lamps in Prob. 17 if the resistance of the line were only 0.12 ohm?

19. An iron wire 1/16 in. in diameter is to be formed into a coil and used as a heater. How many feet will be needed to make a 6-ohm heater?

20. What will be the diameter of an iron wire 100 ft. long which has the same resistance as a copper wire 100 ft. long whose diameter is 0.327 in.?

21. What size aluminum wire 100 ft. long will have the same resistance as the copper wire in Prob. 20?

22. What voltage will be required to force 20 amp. through 300 ft. of iron wire 1/16 in. in diameter?

23. What voltage will be required to force 20 amp. through 300 ft. of copper wire 1/16 in. in diameter?

24. What will be the voltage drop between a generator and a group of lamps which draw 36 amp., if the copper line wires used are each 1,500 ft. long and 1/4 in. in diameter?

51. Current-carrying Capacity of Conductors.—Conductors carrying an electric current become hot if the current which they carry is too great. The National Board of Fire Underwriters has certain requirements regarding the size of wire used for interior wiring. The answer to every problem which asks for the size of wire to be used in a given installation should be checked with the table of safe carrying capacities of wires given on page 68. If the size of wire calculated to be satisfactory for a given voltage drop does not meet the requirements

TABLE OF ALLOWABLE CARRYING CAPACITIES OF WIRES

B & S gage number	Diameter of solid wire, mils	Area, circular mils	Table A. Rubber insulation, amperes	Table B. Other insulation, amperes
18	40.3	1,624	3	5
16	50.8	2,583	6	10
14	64.1	4,107	15	20
12	80.8	6,530	20	25
10	101.9	10,380	25	30
8	128.5	16,510	35	50
6	162.0	26,250	50	70
5	181.9	33,100	55	80
4	204.3	41,740	70	90
3	229.4	52,630	80	100
2	257.6	66,370	90	125
1	289.3	83,690	100	150
0	325.	105,500	125	200
00	364.8	133,100	150	225
000	409.6	167,800	175	275
		200,000	200	300
0000	460	211,600	225	325
		300,000	275	400
		400,000	325	500
		500,000	400	600
		600,000	450	680
		700,000	500	760
		800,000	550	840
		900,000	600	920
		1,000,000	650	1,000
		1,100,000	690	1,080
		1,200,000	730	1,150
		1,300,000	770	1,220
		1,400,000	810	1,290
		1,500,000	850	1,360
		1,600,000	890	1,430
		1,700,000	930	1,490
		1,800,000	970	1,550
		1,900,000	1,010	1,610
		2,000,000	1,050	1,670

1 mil = 0.001 in.

of this table, a larger size of wire must be specified. The two factors, allowable voltage drop and safe carrying capacity, must always be kept in mind when calculating wire sizes.

52. Commercial Wire Sizes.—The sizes of wire listed in the foregoing table are commercial sizes, with the exception of Nos. 3 and 5, which are not standard commercial-wire sizes. When the circular-mil area of the wire needed for an installation has been calculated, refer to this table. The calculated area will very likely fall between two values listed in the table, and the wire size corresponding to the larger of these two sizes should be specified.

Example 8. How many amperes can a 2-wire copper line transmit over a distance of 125 ft. with a line drop of 4 volts? Size 6 B & S gage wire is used.

Solution: The circular-mil area of No. 6 wire is given in the table as 26,250 cir. mils. Therefore, d^2 in the resistance formula equals 26,250. Since the line is 125 ft. long, the total length of wire required is 250 ft. Using these values, we calculate the resistance of the line.

$$R = \frac{10.8(250)}{26,250}$$

Cancelling gives

$$R = \frac{3.6}{35} = 0.103 \text{ ohm}$$

Substituting for E and R in Ohm's law gives

$$\begin{aligned} 4 &= I(.103) \\ 0.103I &= 4 \\ I &= 38.8 \text{ amp. } \textit{Ans.} \end{aligned}$$

Example 9. A group of lamps is connected to a panel box located 150 ft. from the switchboard. The e.m.f. at the switchboard is 115 volts, and the voltage at the panel box should be 112. What size copper wire should be specified for use between switchboard and panel box if the lamps draw 65 amp.? A two-wire system is used.

Solution: The allowable loss in voltage between switchboard and panel box is 3 volts. Using this voltage drop and the current, we calculate the resistance of the line by substituting in Ohm's law.

$$\begin{aligned} 3 &= 65R \\ 65R &= 3 \\ R &= 0.0462 \text{ ohm} \end{aligned}$$

Substitute this resistance value in the formula given in Sec. 50 and solve for d^2.

$$\frac{0.0462}{1} = \frac{10.8(300)}{d^2}$$

Cross-multiplying gives

$$0.0462d^2 = 10.8(300)$$
$$d^2 = \frac{10.8(300)}{0.0462}$$
$$d^2 = \frac{3240}{0.0462}$$
$$d^2 = 70{,}130 \text{ cir. mils}$$

From the table given in Sec. 51, we find that this size is between B & S gages 1 and 2. We should, therefore, specify No. 1 wire. This size will carry as much as 100 amp. and can be safely used.

Example 10. What size wire should be used to wire a 10-hp. motor, if the line voltage is 110?

Solution:

$$10 \text{ hp.} = 7{,}460 \text{ watts}$$
$$W = EI$$
$$7{,}460 = 110I$$
$$110I = 7{,}460$$
$$I = 67.8 \text{ amp.}$$

From the table of safe carrying capacities of wires we find that No. 4 wire must be used.

Problems

1. What size wire should be used for a 5-hp. motor which is to be connected to a 110-volt line?

2. What size wire should be used for a 10-hp., 220-volt motor?

3. A 40-hp. motor is to be used on a 440-volt line. What size wire should be used to make the connections?

4. A circuit consists of twelve 75-watt lamps in parallel. What should be the size of the feeder if the e.m.f. is 110 volts?

5. How many amperes can two copper line wires 0.265 in. in diameter transmit over a distance of 600 ft. with a line drop of 5 volts?

6. How many amperes can two No. 6 copper wires transmit over a distance of 1,200 ft. with a drop of 4 volts?

7. What size copper wire would be required in Prob. 6 if the line drop could be only 2.5 volts and the wires carried 50 amp.?

8. How far can a pair of No. 2 copper wires transmit 45 amp. with a drop of 1.5 volts?

9. How far can two No. 0000 wires transmit 200 amp. with a drop of 5.5 volts?

10. A group of lamps is 250 ft. from a generator. The lamps draw 50 amp., and No. 6 copper wire is used for the line. If the generator voltage is 115-volts, what is the voltage at the lamps?

11. A group of lamps located 400 ft. from a generator require 75 amp. The generator e.m.f. is 114.5, and the voltage at the lamps should be 112. What size copper wire should be used between generator and lamps?

12. What size wire should you specify in Prob. 11, if the distance between generator and lamps were 550 ft. and the e.m.f. at the lamps were to be kept at 110 volts?

13. How many amperes can a pair of No. 0 aluminum wires transmit for 1 mile with a drop of 10 volts?

14. A certain shop, situated 1 mile from a generating station, wishes to use 16 kw. at 220 volts. The e.m.f. at the station is 225 volts. What size copper wire should be used for the 2-wire line?

15. How far can 25 amp. be transmitted over a pair of No. 8 copper line wires with a drop of 7 volts?

16. If the line wires of Prob. 15 were 5/8 in. aluminum wires, how far could 25 amp. be delivered with the same line drop?

17. A pair of No. 000 copper wires transmit 70 amp. over a distance of 1,500 ft. How many volts are lost in the line?

18. How much current can the wires of Prob. 17 transmit over a distance of 1,200 ft. with a line drop of 4.5 volts?

19. A generator supplies a group of 180 lamps each of which requires 0.52 amp. The distance between the generator and the lamps is 260 ft. and the e.m.f. lost in the line should not exceed 3 volts. What size copper wire would you use?

20. If the lamps in Prob. 19 were only 120 ft. from the generator and No. 4 wire were used, would that size be satisfactory?

21. A 220-volt generator is delivering power to three motors connected to the same panel box. One motor draws 7 kw., another 4 kw., and the third 5 kw. The e.m.f. at the panel box is to be 215 volts. What size copper wire should be used if the distance between the generator and the panel box is 200 ft.?

22. What size wire would you use in Prob. 21 if the generator voltage were 440 and the e.m.f. at the panel box were to be kept at 435 volts?

23. A No. 0 copper wire, 1,250 ft. long, carries 125 amp. What is the voltage lost in the wire?

24. What is the voltage drop in an aluminum line carrying 125 amp., if the total length of wire in the line is 1,250 ft. and No. 0 wire is used?

25. A 2-wire copper line is carrying 35 amp. What is the voltage drop per 100 ft. of line, No. 8 being the size of wire used?

26. Suppose that we change the line wires of Prob. 25 to No. 6 aluminum. What will be the voltage drop per 100 ft. of the line?

CHAPTER XI

MULTIPLICATION AND DIVISION OF POLYNOMIALS

53. Multiplication of Polynomials.—When multiplying two polynomials, observe the following rules:

1. *Multiply each term of the first polynomial by the first term of the second polynomial.*

2. *Multiply each term of the first polynomial by each of the remaining terms of the second polynomial, taking one at a time, and arrange the similar terms in vertical columns.*

3. *Add the similar terms to obtain the product.*

Example 1. Multiply as indicated $(3x - 12)(5x + 7)$.

Solution:

$$\begin{array}{rrrl}
3x & -12 & & \\
5x & +7 & & \\
15x^2 & -60x & & \text{multiplying by } 5x \\
 & +21x & -84 & \text{multiplying by } +7 \\
\hline
15x^2 & -39x & -84 & \text{adding similar terms}
\end{array}$$

Check: Let $x = 2$

Then

$$3x - 12 = 3(2) - 12 = 6 - 12 = -6$$

and

$$5x + 7 = 5(2) + 7 = 10 + 7 = 17$$

The product of -6 and 17 equals -102.
Therefore, when 2 is substituted for x in the product $15x^2 - 39x - 84$ obtained above, the result should be -102.

$$\begin{aligned}
15x^2 - 39x - 84 &= 15(2)^2 - 39(2) - 84 \\
&= 15(4) - 78 - 84 \\
&= 60 - 78 - 84 \\
&= -102
\end{aligned}$$

The results agree, and the answer obtained is correct.

Example 2. Multiply $4x^2 - 3xy + 8y^2$ by $3x + 5$.

Solution:

$$\begin{array}{rrrr}
4x^2 & -\ 3xy & +\ 8y^2 & \\
3x & +\ 5y & & \\
\hline
12x^3 & -\ 9x^2y & +\ 24xy^2 & \\
 & +\ 20x^2y & -\ 15xy^2 & +\ 40y^3 \\
\hline
12x^3 & +\ 11x^2y & +\ 9xy^2 & +\ 40y^3
\end{array}$$

Check: Let $x = 2$, $y = 2$

$$\begin{aligned} 4x^2 - 3xy + 8y^2 &= 4(2)^2 - 3(2)2 + 8(2)^2 \\ &= 4(4) - 12 + 8(4) \\ &= 16 - 12 + 32 \\ &= 36 \\ 5x + 7 &= 5(2) + 7 = 10 + 7 = 17 \end{aligned}$$

The product of 36 and 17 is 576.
Substitute $x = 2$, $y = 2$ in the product obtained.

$$\begin{aligned} 12x^3 + 11x^2y + 9xy^2 + 40y^3 &= 12(2)^3 + 11(2)^2 2 + 9(2)(2)^2 + 40(2)^3 \\ &= 12(8) + 11(4)2 + 9(2)4 + 40(8) \\ &= 96 + 88 + 72 + 320 \\ &= 576 \end{aligned}$$

Problems

Multiply each of the following as indicated, and check Prob. 1 to 15:

1. $(x - 1)(x + 1)$
2. $(x + y)(x + y)$
3. $(6x + 49)(3x - 2y)$
4. $(7x - 2y)(8x - 5y)$
5. $(3x - 5)(4x + 7)$
6. $(5x^2 - 6y^2)(4x - 7y)$
7. $(8x^2 - 7y^2)(8x^2 + 7y^2)$
8. $(10x - 13y)(20x + 3y)$
9. $(19x + 17y)(23x - 7y)$
10. $(15a - 13b)(2a + 9b)$
11. $(9x^2 - 4)(8x^2 + 15)$
12. $(17x^2 - 25y^2)(17x^2 + 25y^2)$
13. $(5x^2 + 8y^2)(5x^2 - 8y^2)$
14. $(4a^2 - 3b^2)(5a^2 + 8b^2)$
15. $(11a^2 + 7b^2)(9a^2 - 14b^2)$
16. $(ax - by)(ax - by)$
17. $(5x - 7y)^2$
18. $(ax - by)^2$
19. $(ax + by)^2$
20. $(9x^2 - 4x + 1)(3x + 7)$
21. $(x^4 - x^2y^2 + y^4)(x^2 + y^2)$
22. $(25x^2 - 12xy + 15y^2)(11x - 12y)$
23. $(37a^2 + 15ab - 28b^2)(13a - 20b)$
24. $(21a^2 + 7a + 56)(7a - 4)$
25. $(9x^2 - 8y^2)(5x^3 - 4y^3)$
26. $(4x^2 - 3x + 1)^2$
27. $(3x^2 + 5x - 6)(x^2 - x + 4)$
28. $(4a^2 - 7ab + 5b^2)(3a^2 + 5ab - 7b^2)$
29. $(4x^2 - 5x + 7)(-3x^2 - 4x + 8)$
30. $(5x^2 - 7x + 9)^2$
31. $(8a^2b - 7ab^2 - 4b^3)(6a^2 - 5ab + 9b^2)$
32. $(13x^2y + 15xy^2 + 8y^3)(9x^2 - 13xy - 10y^2)$
33. $(9x^2y + 18xy^2 - 5y^3)(6x^2 - 7xy - 12y^2)$
34. $(25a^2b - 35ab^2 - 9b^3)(15a^2 + 17ab + 2b^2)$
35. $(4c^2d + 11cd^2 - 13d^3)(16c^2 - 15cd + 4d^2)$
36. $(11x^2y - 22xy^2 + 33y^3)(11x^2 + 5xy - 5y^2)$

54. Division of Polynomials.—When dividing one polynomial by another, arrange both the dividend and the divisor according to the descending powers of the same letter and then perform exactly the same operations as are performed in long division in arithmetic. To obtain the terms of the quotient, always divide the first term of the dividend by the first term of the divisor.

Example 3. Divide $x^2 + 3x - 18$ by $x - 3$.

Solution:

$$\begin{array}{r|l} & x \quad + 6 \ \textit{Ans.} \\ \hline x - 3 & x^2 + 3x - 18 \\ & \underline{x^2 - 3x} \\ & \quad + 6x - 18 \\ & \quad \underline{+ 6x - 18} \end{array}$$

The division is performed as follows:

1. Divide x^2 by x. This gives the first term in the quotient, which is x.
2. Multiply $x - 3$ by x, the first term of the quotient, and arrange the terms obtained under the similar terms in the dividend.
3. Subtract the product obtained in step 2 from the dividend and bring down the next term of the dividend. This gives the remainder $6x - 18$.
4. Divide $6x$ by x, the first term of the divisor. The result 6 forms the second term of the desired quotient.
5. Multiply $x - 3$ by 6, place the terms obtained under the remainder, and subtract.
6. Since the result of this subtraction is zero, $x + 6$ is the exact quotient.

Check: Let $x = 2$

Then

$$x^2 + 3x - 18 = (2)^2 + 3(2) - 18 = 4 + 6 - 18 = -8$$

and

$$x - 3 = 2 - 3 = -1$$

The quotient of -8 divided by -1 is 8.

When 2 is substituted in the answer obtained, the result should be 8.

$$x + 6 = 2 + 6 = 8$$

Example 4. Divide $7x^2y^2 - 15x^3y - 25xy^3 + 6x^4 - 5y^4$ by $2x^2 - y^2 - 5xy$

Solution: 1. Arrange the two expressions according to the descending powers of x.

This gives

$$6x^4 - 15x^3y + 7x^2y^2 - 25xy^3 - 5y^4$$

and

$$2x^2 - 5xy - y^2$$

2. Divide, following the order given in Ex. 1.

$$\begin{array}{r|l}
 & 3x^2 + 5y^2 \ \textit{Ans.} \\
\hline
2x^2 - 5xy - y^2 & 6x^4 - 15x^3y + 7x^2y^2 - 25xy^3 - 5y^4 \\
 & \underline{6x^4 - 15x^3y - 3x^2y^2} \\
 & \qquad\qquad + 10x^2y^2 - 25xy^3 - 5y^4 \\
 & \qquad\qquad \underline{+ 10x^2y^2 - 25xy^3 - 5y^4}
\end{array}$$

Check: Let $x = 2,\ y = 2$

$$\begin{aligned}
6x^4 - 15x^3y + 7x^2y^2 - 25xy^3 - 5y^4 &= 6(2)^4 - 15(2)^3 2 + 7(2)^2(2)^2 \\
&\qquad - 25(2)(2)^3 - 5(2)^4 \\
&= 6(16) - 15(8)2 + 7(4)4 \\
&\qquad - 25(2)8 - 5(16) \\
&= 96 - 240 + 112 - 400 - 80 \\
&= -512 \\
2x^2 - 5xy - y^2 &= 2(2)^2 - 5(2)(2) - (2)^2 \\
&= 2(4) - 20 - 4 \\
&= 8 - 20 - 4 \\
&= -16
\end{aligned}$$

The quotient of -512 divided by $-16 = 32$.

$$\begin{aligned}
3x^2 + 5y^2 &= 3(2)^2 + 5(2)^2 \\
&= 3(4) + 5(4) \\
&= 12 + 20 \\
&= 32
\end{aligned}$$

The results agree.

Example 5. Divide $x^4 - y^4$ by $x + y$.

Solution: When arranging these expressions for the division, leave a space between the x and $-y$ large enough so that three or four terms could be written between them. Then perform the division.

$$\begin{array}{r|l}
 & x^3 - x^2y + xy^2 - y^3 \ \textit{Ans.} \\
\hline
x + y & x^4 \qquad\qquad\qquad\qquad\qquad -y^4 \\
 & \underline{x^4 + x^3y} \\
 & \quad - x^3y \\
 & \quad \underline{-x^3y - x^2y^2} \\
 & \qquad\qquad +x^2y^2 \\
 & \qquad\qquad \underline{+x^2y^2 + xy^3} \\
 & \qquad\qquad\qquad -xy^3 - y^4 \\
 & \qquad\qquad\qquad -xy^3 - y^4
\end{array}$$

Check: Let $x = 3$, $y = 2$

$$
\begin{aligned}
x^4 - y^4 &= (3)^4 - (2)^4 = 81 - 16 = 65 \\
x + y &= 3 + 2 \qquad\quad = 5
\end{aligned}
$$

The result of 65 divided by 5 is 13.

$$
\begin{aligned}
x^3 - x^2y + xy^2 - y^3 &= (3)^3 - (3)^2(2) + 3(2)^2 - (2)^3 \\
&= 27 - 9(2) + 3(4) - 8 \\
&= 27 - 18 + 12 - 8 \\
&= 13
\end{aligned}
$$

The results agree.

Problems

Arrange each of the following in proper order; then divide the quantity in the first column by the quantity in the second column:

	Divide	By
1.	$x^2 + 24 + 10x$	$x + 4$
2.	$x^2 - 35 + 2x$	$x + 7$
3.	$140 + x^2 - 24x$	$x - 14$
4.	$30x^2 - 84 - 37x$	$6x + 7$
5.	$13x - 44 + 15x^2$	$3x - 4$
6.	$40x^2 - 26 - 49x$	$2 + 5x$
7.	$16xy + 9x^2 - 4y^2$	$x + 2y$
8.	$6a^2 - 15b^2 + ab$	$2a - 3b$
9.	$11xy - 14y^2 + 15x^2$	$7y + 5x$
10.	$10x^2y^2 + 75x^4 - 176y^4$	$5x^2 + 8y^2$
11.	$6x^4 + 40y^4 + 24x^3y + 10xy^3$	$2x + 8y$
12.	$24a^3 - 15ab^2 + 35b^3 - 56a^2b$	$3a - 7b$
13.	$15a^2b - 12ab^2 - 18b^3 + 10a^3$	$5a^2 - 6b^2$
14.	$18a^3 - 15a^2b^2 + 24ab - 20b^3$	$6a - 5b^2$
15.	$95ab - 85b^3 + 76a^3 - 68a^2b^2$	$4a^2 + 5b$
16.	$12x^4 - 17x^3 - 14x^2 + 13x - 8$	$3x^2 - 2x + 1$
17.	$-47x^3 + 37x^2 + 12x^4 - 42 - 26x$	$7 - 5x + 4x^2$
18.	$11x^2 + 56 - 19x + 6x^3$	$3x^2 + 8 - 5x$
19.	$356x - 230 - 211x^2 + 40x^3$	$8x - 23$
20.	$44xy^2 - 40x^2y + 15x^3 - 16y^3$	$3x - 2y$
21.	$11a^2b + 39b^3 + 120a^3 - 125ab^2$	$15a^2 - 13b^2 + 7ab$
22.	$-64a^2b^2 - 70b^4 - 34a^3b + 143a^4 - 113ab^3$	$13a^2 - 9ab - 10b^2$
23.	$60a^6 - 66b^6 - 8a^4b^2 - 151a^2b^4$	$5a^2 + 6b^2$
24.	$48x^5 - 15x^2 - 38x^3 + 25 - 70x$	$3x^2 - 5$
25.	$40xy^3 - 24x^3y + 42x^4 - 82x^2y^2 + 20y^4$	$3x^2 - 5y^2$
26.	$116x^9 - 17x^3y^6 + 40y^9 - 157x^6y^3$	$4x^3 - 5y^3$
27.	$35x^6 + 5 - 22x^2 - 4x^4$	$5x^4 + 3x^2 - 1$
28.	$148x^6 - 44x^4 + 96x^8 - 40x^2$	$32x^6 - 4x^4 - 8x^2$
29.	$x^3 + 1$	$x + 1$
30.	$x^3 + y^3$	$x^2 - xy + y^2$

31.	$a^4 + a^2b^2 + b^4$	$a^2 + ab + b^2$
32.	$x^3 - y^3$	$x - y$
33.	$343x^3 - 512y^3$	$7x - 8y$
34.	$27a^3 - 125b^6$	$9a^2 + 15ab^2 + 25b^4$
35.	$a^3 + b^3$	$a + b$
36.	$6x^3 - 5x^2y + 10xy^2 - 24y^3$	$2x - 3y$
37.	$4x^3 - 6x^2y + 6y^3 + 8xy^2$	$2x + y$
38.	$12xy^3 + 16x^2y^2 - 15y^4 - 20x^3y + 15x^4$	$3x^2 - 4xy + 5y^2$
39.	$60x^6 - 126x^4y^2 + 48y^6 + 18x^2y^4$	$5x^2 - 8y^2$
40.	$-8x - 45x^3 + 60x^8 - 25x^5 + 12x^6$	$12x^6 - 8x - 5x^3$

CHAPTER XII

ANGLES, AREAS, AND VOLUMES

55. Angles.—The figure formed by two straight lines which proceed from the same point is called an "angle." The point from which the two lines proceed is called the *vertex* of the angle.

56. Comparison of Angles.—Suppose angle A of Fig. 24 to be placed over angle B so that the horizontal sides of the two angles coincide and point A falls on point B. It will be

FIG. 24.—Types of angles.

seen that the slanting side of angle A falls to the right of the slanting side of angle B. Angle A is said to fall within angle B and is smaller than angle B.

If the slanting sides of the two angles had coincided exactly, the two angles would have been equal.

If the slanting side of angle A had fallen to the left of the slanting side of angle B, angle A would be larger than angle B. This would be the case if angle C or angle D (Fig. 24) were placed upon angle B with their vertices and horizontal sides coinciding.

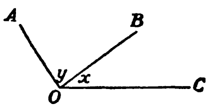


FIG. 25.—Adjacent angles.

57. Adjacent Angles.—Two angles which have one side in common and the same vertex are *adjacent angles*. In Fig. 25, x and y are adjacent angles.

The notation for angle x is BOC or COB, and for angle y it is AOB or BOA. The middle letter always denotes the vertex of the angle.

58. Perpendiculars.—When two lines meet so that the adjacent angles formed are equal, then the lines are said to be perpendicular, one to the other.

59. Right, Acute, and Obtuse Angles.—The angles formed by two lines which are perpendicular to each other are *right angles*. In Fig. 26, angles ADC and CDB are right angles.

An angle which is smaller than a right is an *acute angle*. Angles A and B, Fig. 24, are acute angles.

An angle which is larger than a right angle is an *obtuse angle*. Angle D, Fig. 24, is an obtuse angle.

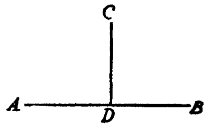


FIG. 26.—Perpendicular lines.

60. Degrees, Minutes, Seconds.—An angle *degree* is one-ninetieth of a right angle. If a degree is divided into 60 equal parts, each is called one *minute*. Each minute may be divided into 60 equal parts each of which is one *second*.

61. Plane Figures.—There are several common types of plane figures with which every student should be familiar. In Fig. 27, we have several of these common figures.

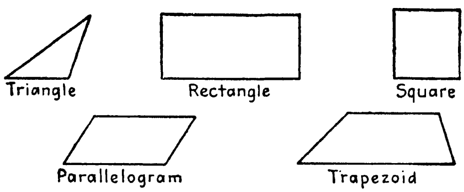


FIG. 27.—Plane figures.

A triangle is a plane figure having three sides.

A *rectangle* is a plane figure of four sides all of whose angles are right angles.

A *square* is a rectangle all of whose sides are of equal length.

A *parallelogram* is a four-sided figure whose opposite sides are parallel.

A *trapezoid* is a four-sided figure two of whose sides are parallel.

62. Sum of the Angles of a Triangle.—If you were to draw a number of triangles and then measured the three interior angles of each triangle with a protractor, you would find the

sum of the angles of each triangle to be about the same. By geometry, we can easily prove that

The sum of the interior angles of a triangle is equal to 180 *deg.*

Use this fact as a basis in solving the following problems:

Problems

1. If the angles of a triangle are represented by x, $2x$, and $3x$, find the numerical value of each.

2. If the three angles of a triangle are equal, how many degrees in each angle?

3. In a right triangle, how many degrees are there in the two acute angles?

4. If one acute angle of a right triangle is 27°, how many degrees in the other acute angle?

5. One angle of a triangle measures 32 and another 105°. How large is the third angle?

6. One angle of a triangle is three times as large as the second and the third is half as large as the second. How many degrees in each angle?

7. Find the angles of a triangle if the first is 26° larger than the second and 5° smaller than the third.

8. Find the three angles of a triangle if the first is $\frac{1}{5}$ of the second, plus 11°, and the third is twice the first, plus 91°.

9. Find the three angles of a triangle if the second is three times the first, minus 21°, and the first is 5° smaller than four times the third.

10. Find the three angles of a triangle if the first is twice the second, minus 42°, and the second is twice the third, less 31°.

63. Linear and Square Units.—A line has length but no width. We can measure the length of a line in inches, feet, or yards. These are *linear units.*

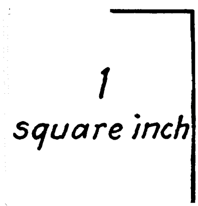


Fig. 28.

Figure 28 shows a square whose side measures one inch. This is called a *unit square.* The surface enclosed by the sides of this square is one square inch.

Other square units are the square foot, the square yard, the square mile, etc.

64. Areas.—The area of a plane figure is the equivalent number of square units contained in the figure. The areas of simple figures are easily determined if we know certain of the dimensions.

The area of a triangle is found by taking one-half the product of the base and altitude.

The area of any rectangular figure is found by multiplying its base by its altitude.

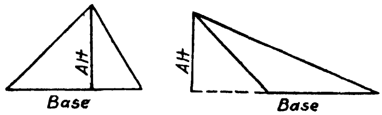


FIG. 29.

To find the area of a parallelogram or a trapezoid, *divide the figure into two triangles and find the sum of the areas of the triangles.*

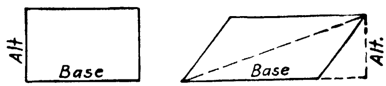


FIG. 30.

This same method can be used to determine the area of a figure of any number of sides if it is possible to determine the base and altitude of each of the triangles and rectangles into which the figure is divided.

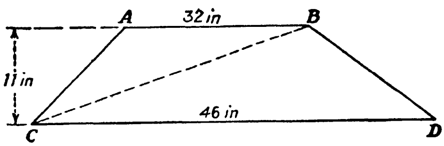


FIG. 31.

Example 1. Find the area of the trapezoid shown in Fig. 31.

Solution: Divide the figure into two triangles by drawing the diagonal CB.

$$\text{Area of triangle } ABC = \tfrac{1}{2}(32)11 \text{ sq. in.} = 176 \text{ sq. in.}$$
$$\text{Area of triangle } CBD = \tfrac{1}{2}(46)11 \text{ sq. in.} = 253 \text{ sq. in.}$$

The area of the trapezoid $ABDC$ is then

$$176 + 253 = 429 \text{ sq. in. } \textit{Ans.}$$

Problems

1. Find the area of a square whose side measures 26 ft. How would you represent the area of a square whose side measures x ft.?

2. Find the area of a triangle whose base is 29 ft. long and whose altitude is 13 ft.

3. The two parallel sides of a trapezoid measure 37 ft. and 63 ft., respectively. Find the area of the trapezoid, if the altitude is 24 ft.

4. Find the area of a trapezoid whose bases measure 75 ft. and 38 ft., respectively. The altitude is 30 ft.

5. The area of a rectangle is 470 sq. ft. and its altitude is 23.5 ft. What is the base of the rectangle?

6. The area of a rectangle is 576 sq. ft. It is 14 ft. long. How wide is it?

65. Circumference and Area of Circles.—The circumference C of a circle is found by the formula

$$C = \pi d$$

π is a constant whose value is 3.1416, d is the diameter of the circle.

The area A of a circle is found using the formula

$$A = \frac{\pi d^2}{4}$$

7. Find the circumference and area of a circle whose diameter is 15 in.

8. Find the circumference and area of a circle whose diameter is 325 ft.

9. Find the circumference and area of a circle whose diameter is 46.7 ft.

10. Find the circumference and area of a circle whose diameter is 52.8 in.

11. The area of a circle is 1,194.59 sq. in. Find its diameter.

12. The area of a circle is 9,676.89 sq. ft. Find its diameter.

13. The area of a circle is 286.521 sq. in. Find its diameter.

14. The area of a circle is 606.987 sq. ft. Find its diameter.

15. The area of a circle is 1,385.44 sq. in. Find its diameter.

16. The circumference of a circle is 128.81 ft. Find its diameter.

17. The circumference of a circle is 1,068.1 ft. Find its diameter

18. The circumference of a circle is 237.5 in. Find its diameter.

19. The circumference of a circle is 116.24 ft. Find its diameter.

20. The circumference of a circle is 970.75 in. Find its diameter.

66. Circular-mil Area of Busbars.—In order to determine the cross-sectional area of a busbar in circular mils, it is necessary to know the relation between the square mil and the circular mil. This is determined as follows:

The area of a circle 1 mil in diameter = 1 cir. mil.

The area of the same circle, expressed in square mils, is $\pi/4$ sq. mils

$$\therefore \frac{\pi}{4} \text{ sq. mils} = 1 \text{ cir. mil}$$

$$\pi \text{ sq. mils} = 4 \text{ cir. mils}$$

$$1 \text{ sq. mil} = \frac{4}{\pi} \text{ cir. mils}$$

$$1 \text{ sq. mil} = 1.2732 \text{ cir. mils}$$

If, therefore, we desire to know the cross-sectional area of a busbar in circular mils, we find its area in square mils and multiply by 1.2732

Example 2. What is the circular-mil area of a copper conductor which is ⅛ in. thick and ½ in. wide?

Solution:

$$\tfrac{1}{8} \text{ in.} = 125 \text{ mils}$$
$$\tfrac{1}{2} \text{ in.} = 500 \text{ mils}$$

$$\begin{aligned} \text{Cross-sectional area of bar} &= 125(500) \text{ sq. mils} \\ &= 62{,}500 \text{ sq. mils} \\ \text{The circular-mil area} &= 62{,}500(1.2732) \\ &= 79{,}575 \text{ cir. mils} \end{aligned}$$

Problems

1. How many circular mils are there in a copper busbar 2 in. wide and ¼ in. thick?

2. How many circular mils are there in a square bar of copper ⅜ in. on a side?

3. What is the circular-mil area of a copper ribbon 1½ in. wide and 1/32 in. thick?

4. A copper busbar measures 4 by ⅜ in. What is its circular-mil area?

5. A busbar is 3 in. wide and ¼ in. thick. How many circular mils does it contain?

6. A busbar whose circular-mil area should be about 1,100,000 cir. mils is to be made of ¼-in. copper. What width of copper bar should be used, if the sizes available vary ½ in.?

7. What size should be used in Prob. 6, if the desired circular-mil area is 1,550,000 cir. mils?

67. Relation between the Sides of a Right Triangle.—In every right triangle a definite relation exists between the sides

of the triangle, so that when the length of two of the sides is known, the length of the third side can be calculated. The side opposite the right angle is termed the *hypotenuse* of the triangle, and the two sides forming the right angle are known as the "legs" of the triangle. The law which states the relation between these sides is as follows:

In any right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.

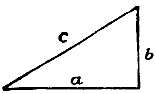


FIG. 32.

If we represent the sides of a right triangle by the letters a, b, and c, as in Fig. 32, the above law may be stated in the form of the equation

$$c^2 = a^2 + b^2$$

This formula is very important and has many applications in electricity. Its use in the solution of problems is illustrated by the two examples which follow.

Example 3. The two legs of a right triangle measure 24 and 10 ft., respectively. Find the length of the hypotenuse.

Solution: Let $a = 24$ ft., $b = 10$ ft.

Then

$$\begin{aligned} c^2 &= (24)^2 + (10)^2 \\ c^2 &= 576 + 100 \\ c^2 &= 676 \\ c &= 26 \text{ ft. } \textit{Ans.} \end{aligned}$$

Example 4. The hypotenuse of a right triangle is 72. One of the shorter sides is 40. Find the third side.

Solution: $c = 72$, $a = 40$

$$\begin{aligned} (72)^2 &= (40)^2 + b^2 \\ 5{,}184 &= 1{,}600 + b^2 \\ 5{,}184 - 1{,}600 &= b^2 \\ 3{,}584 &= b^2 \\ b^2 &= 3{,}584 \\ b &= 59.87 \textit{ Ans.} \end{aligned}$$

Problems

1. The hypotenuse of a right triangle is 39. One of the shorter sides is 15. Find the third side.

2. What is the length of the hypotenuse of a right triangle whose other sides measure 72 ft. and 96 ft.?

3. A rectangle is 48 ft. long and 20 ft. wide. Find the length of its diagonal.

4. A regulation baseball diamond is laid out in the form of a square each side of which measures 90 ft. What is the distance from home plate to second base?

5. The diagonal of a square is 13 in. long. How long is each side?

In each of the following problems a, b, and c represent the sides of a right triangle, c being the hypotenuse.

6. $a = 22.5$, $b = 54$. Find c.
7. $c = 19.0$, $b = 15.2$. Find a.
8. $c = 36$, $a = 28.8$. Find b.
9. $c = 35.1$, $a = 32.4$. Find b.
10. $a = 35$, $b = 26$. Find c.
11. $c = 96$, $b = 44$. Find a.
12. $c = 2.3$, $b = 1.38$. Find a.
13. $c = 0.65$, $a = 0.25$. Find b.
14. $c = 0.015$, $b = 0.009$. Find a.

68. Volumes.—The volume of a solid of uniform cross-section is obtained by multiplying its cross-sectional area by its length.

Use this rule in solving the following problems:

Problems

1. Find in cubic inches the volume of a steel bar 3 ft. long, 3 in. wide, and $2\frac{1}{2}$ in. thick.

2. Find the volume of a cylinder 18 in. in diameter and 38 in. long.

3. The inside dimensions of a steel drum are 2 ft. 6 in. in diameter by 3 ft. 6 in. deep. How many gallons of oil will it hold? There are 231 cu. in. to a gallon.

4. A copper busbar is 3 in. wide, $\frac{3}{8}$ in. thick, and 10 ft. long. How much does it weigh if copper weighs 0.3184 lb. per cubic inch?

5. Find the weight of a copper rod $3\frac{1}{2}$ in. in diameter and 14 ft. long.

6. The inside dimensions of a steel drum are 35 in. in diameter and 44 in. deep. How many gallons of oil will it hold?

7. Find the weight of a copper rod $4\frac{1}{2}$ in. in diameter and 18 ft. long.

8. The inside dimensions of a tank are 11 ft. in diameter and 20 ft. deep. How many barrels can the tank contain if there are 63 gal. to the barrel?

Review Equations

Solve and check each of the following equations:

1. $\frac{x + 4}{3} = \frac{x - 5}{4}$

2. $\frac{x - 4}{3} = \frac{4 - x}{5}$

3. $x - \frac{12 - x}{7} = 4$

4. $3x - \frac{4x - 3}{5} = \frac{4}{3}$

5. $\frac{2x + 10}{3} + \frac{3 - x}{9} = 2$

6. $2 - \frac{3x + 4}{5} = 7 + \frac{2x - 6}{3}$

7. $3 + \frac{5x - 9}{4} = 15 - \frac{6x + 13}{10}$

8. $3 - \frac{4(y - 2)}{3} = \frac{2(2y - 1)}{6} - \frac{3y + 1}{4}$

9. $5x - \frac{2x + 13}{3} + \frac{7x - 51}{4} = 62$

10. $\frac{3x + 26}{5} - \frac{7 - 2x}{3} - 10 = \frac{5x - 16}{3} - 8\frac{(2x - 1)}{15} + 13$

11. $\frac{5x - 2}{4} + 2 = x - \frac{6x - 8}{2}$

12. $\frac{10y + 3}{3} - \frac{6y - 7}{2} = 10(y - 1)$

13. $\frac{3x + 5}{7} - \frac{2x + 7}{3} + 10 - \frac{3x}{5} = 0$

14. $\frac{5x - 3}{7} - \frac{9 - x}{3} = \frac{5x}{2} + \frac{19}{6}(x - 4)$

15. $\frac{7y + 9}{8} - \frac{3y + 1}{7} = \frac{9y - 13}{4} - \frac{249 - 9y}{14}$

16. $\frac{9x + 20}{36} = \frac{4(x - 3)}{5x - 4} + \frac{x}{4}$

17. $\frac{10x + 17}{18} - \frac{12x + 2}{13x - 16} = \frac{5x - 4}{9}$

CHAPTER XIII

FACTORING. SOLUTION OF QUADRATIC EQUATIONS

69. Factors.—The factors of any number or any quantity are the numbers which, when multiplied, produce the original number or quantity. For example, 3 multiplied by 2 gives 6. The factors of 6 are, therefore, 3 and 2.

A *prime factor* is a quantity which has no other factors but itself and one. Now, 2 and 6 are factors of 12. However, 6 is not a prime factor; the prime factors of 12 are 2, 2, and 3.

Factoring is the process of separating a product into its prime factors.

We shall consider three cases of factoring and shall examine first the products formed when certain factors are multiplied. After studying these products with relation to their factors, we shall be able readily to recognize the products of each type and shall have little difficulty in determining the factors for each case.

70. Monomial Factors.—One of the simplest products formed in algebra is that of a polynomial and a monomial. Let us study a few products of this kind.

$$4x^2y^2(a + b) = 4ax^2y^2 + 4bx^2y^2$$

Study the product $4ax^2y^2 + 4bx^2y^2$.

What numerical factors appear in each term?

What literal factors appear in each term?

What is the lowest power of each of these literal factors in any term?

You will note that 4 is the numerical factor; x and y are the literal factors common to each term. The lowest power in which each occurs in any term is the second.

Therefore, $4x^2y^2$ is common to both terms and is the largest monomial factor of $4ax^2y^2 + 4bx^2y^2$. The other factor is

obtained by mentally dividing $4ax^2y^2 + 4bx^2y^2$ by $4x^2y^2$. This gives $a + b$.

We indicate that these are the factors by writing:

$$4ax^2y^2 + 4bx^2y^2 = 4x^2y^2(a + b)$$

It is not necessary to reduce $4x^2y^2$ to the prime factors 2, 2, x, x, y, and y. A monomial factor need not be separated in this manner. All other factors must be reduced to prime factors.

Determine mentally the following products and study them:

1. $6a(a + b)$
2. $3x(x + y)$
3. $4x^2y(x^2 - xy)$
4. $ab(ax + by + cz)$
5. $x^2y(a^2x + ay + bxy)$
6. $4axy(5a^2x - 6aby + 3b^2y)$

Example 1. Factor $15a^2b - 10a^3b^2c + 20a^3b^2c^4$.

Solution: 5 is the numerical factor common to each term. The letters a and b appear in each term, and the lowest power of a in any term is a^2, the lowest power of b is b. Therefore, $5a^2b$ is the monomial factor.

$$\therefore 15a^2b - 10a^3b^2c + 20a^3b^2c^4 = 5a^2b(3 - 2abc + 4abc^4)$$

Problems

Write the factors of each of the following:

1. $6a^2 + 6ab$
2. $14a^2x + 7ay$
3. $3x^2 + 3xy$
4. $14a^3x^2 - 21a^2xy$
5. $4x^4y - 4x^3y^2$
6. $36a^2b^2c + 12ab^2c^2 - 18a^3bc^3$
7. $abcx + a^2c^2x^2 - ab^2x$
8. $a^2bx + ab^2y + abcz$
9. $a^2x^3y + ax^2y^2 + bx^3y^2$
10. $20a^3x^2y - 24a^2bxy^2 + 12ab^2xy^2$
11. $12x^3y + 20x^2y - 4xy$
12. $15x^5y^2 + 3x^2y^2 - 9x^4y^3$
13. $3x^4 - 4x^3 - x^2 + 5x$
14. $22x^3y^2 - 44x^2y^3 + 33x^2y - 55x^4y$
15. $24a^4b^4c^2 + 32a^3b^5c^2 - 16a^4b^4c^3 - 8a^2b^3c$

71. Product of Two Binomials.—The product of two binomials is usually a trinomial, except when the sum of the cross-products is zero. The product of two binomials can readily be determined mentally, as explained in the examples which follow:

Example 2. Mentally obtain the product of $x + 3$ and $x + 4$.

Solution: 1. Multiply the first term of the first binomial by the first term of the second. This gives x^2.

2. Multiply the first term of each binomial by the second term of the other, and add these two products. This gives $3x + 4x = 7x$.

3. Multiply the last term of the first binomial by the last term of the second. This gives $+12$.

4. The complete product is $x^2 + 7x + 12$.

Example 3. Mentally obtain the product of $3x + 2y$ and $2x - 5y$.

Solution: Proceed as explained in the preceding example. The result of performing the operations in the three steps is

$$6x^2, \ (-15xy + 4xy), \ -10y^2$$

The complete product is $6x^2 - 11xy - 10y^2$.

Write the result of the following multiplications, doing the work mentally:

1. $(x + 5)(x + 3)$
2. $(y + 3)(y - 4)$
3. $(2x + 3)(2x - 5)$
4. $(x + y)(2x + y)$
5. $(x + y)(2x - y)$
6. $(2x - 5)(3x - 4)$
7. $(3x - 5)(2x - 7)$
8. $(4x + 7)(5x - 9)$
9. $(3x - 2)(2x + 7)$
10. $(3x - 2)(2x - 7)$
11. $(3x - 4y)(4x + 3y)$
12. $(3x + 4y)(4x - 3y)$
13. $(5x - 7y)(2x - 3y)$
14. $(7x + 4y)(8x - 5y)$
15. $(10x - 3y)(2x + 9y)$
16. $(3x + 1)(2x - 4)$
17. $(5y - 1)(2y + 1)$
18. $(3x - 4y)(5x - 6y)$
19. $(4a - 5b)(7a - 8b)$
20. $(9a - 2b)(3a - 4b)$

In studying the above products, take particular notice of the signs of the second and third terms of the trinomials formed. First look at the sign of the third term and then at the sign of the second term.

In Prob. 1, what is the sign of the third term; of the second term?

In Prob. 2, what is the sign of the third term; of the second term?

In Prob. 6, what is the sign of the third term; of the second term?

You will notice that if the third term is positive, the two factors which were multiplied to produce that term had the same sign. Notice, also, that the sign of these two factors is the same as the sign of the second term. If the third term is negative, one of the factors of this term must be positive and

the other negative. In this case, the second term may be either positive or negative.

Trinomials are factored by trial. From the facts noted in the preceding paragraph, we can deduce the following statements, which will help you in factoring if you keep them in mind.

1. If the third term of a trinomial is positive, each of its binomial factors will have its terms separated by the same sign. This sign is the same as the sign of the second term of the trinomial.

2. If the third term of a trinomial is negative, one of its binomial factors will have its terms separated by a plus sign and the other will have its terms separated by a minus sign.

Any trinomial having binomial factors may be factored by trial. Two examples are given to illustrate the method:

Example 4. Factor $x^2 - 2x - 8$.

Solution: 1. Select two factors of the first term. In this case, they can be only x and x.

2. Select two factors of the third term. Suppose that we try -8 and 1. Since the middle term of the trinomial is negative, the larger of these two factors is taken as the negative one.

3. Combine the factors of the first two steps to form two binomials, selecting for each binomial one factor from step 1 and one from step 2. This gives the factors $x - 8$ and $x + 1$.

4. Mentally multiply these two factors. Their product is $x^2 - 7x - 8$. The factors chosen are incorrect.

5. Choose two different factors for the third term of the trinomial; try -4 and 2.

6. This gives the binomial factors $x - 4$ and $x + 2$. Multiplying these two gives $x^2 - 2x - 8$. Therefore, these are the correct factors of $x^2 - 2x - 8$, and we indicate this by writing

$$x^2 - 2x - 8 = (x - 4)(x + 2)$$

Example 5. Factor $12x^2 - x - 20$.

Solution: 1. Try $6x$ and $2x$ as factors of $12x^2$, and -10 and 2 as factors of -20, and write the binomial trial factors $6x - 10$ and $2x + 2$.

2. Multiplying these gives $12x^2 - 8x - 20$. Incorrect.

3. Try $4x$ and $3x$ as factors of $12x$. The binomials to be multiplied are $4x - 10$ and $3x + 2$.

4. Multiplying these gives $12x^2 - 22x - 20$. Incorrect.

5. Using $4x + 2$ and $3x - 10$ gives a product of $12x^2 - 36x - 20$. Incorrect.

6. Try 5 and 4 as factors of 20. Write the factors $3x - 5$ and $4x + 4$. Multiplying these gives $12x^2 - 8x - 20$. Incorrect.

7. Reverse the factors 4 and 5, thus obtaining the factors $3x - 5$ and $4x + 5$. Multiplying these gives $12x^2 - x - 20$. Correct.

$$\therefore 12x^2 - x - 20 = (3x - 4)(4x + 5)$$

The student will find, after he has had some experience in factoring by trial, that he will, in most cases, be able to determine the correct factors after two or three trials.

Problems

Factor each of the following by trial. Check each odd-numbered problem by multiplying the factors:

1. $x^2 - 8x + 12$
2. $x^2 - 3x + 2$
3. $x^2 - 6x + 5$
4. $a^2 - 4a + 3$
5. $y^2 - 5y - 6$
6. $4a^2 - 12a + 5$
7. $4x^2 + 8x - 5$
8. $6y^2 - 17y - 14$
9. $6a^2 - 23a + 20$
10. $6x^2 - 25x + 14$
11. $2x^2 + xy - y^2$
12. $x^2 + 2xy + y^2$
13. $12x^2 - 43xy + 35y^2$
14. $10x^2 - 37xy - 36y^2$
15. $25x^2 + 60xy + 36y^2$
16. $25x^2 - 20xy + 4y^2$
17. $56x^2 - 3xy - 20y^2$
18. $8y^2 + 2y - 1$
19. $63a^2 - 35ab + 6b^2$
20. $63a^2 - 156ab - 32b^2$
21. $15y^2 + 7y - 4$
22. $40x^2 + 27xy - 4y^2$
23. $24a^2 + 6ab - 9b^2$
24. $24x^2 - 77xy + 63y^2$
25. $15a^2 + 2ab - 8b^2$
26. $52x^2 + 97xy - 45y^2$
27. $63x^2 - 130xy - 32y^2$
28. $4x^2 - 28xy + 49y^2$

72. The Difference of Two Squares.—When the sum of two numbers is multiplied by the difference of the same two numbers, the result is the difference of the squares of the two numbers. This gives a product of the type $a^2 - b^2$, $4x^2 - y^2$, etc.

Write the following products, doing the work mentally:

1. $(a + b)(a - b)$
2. $(x - 1)(x + 1)$
3. $(2x + 1)(2x - 1)$
4. $(4a + 5b)(4a - 5b)$
5. $(3x - 4y)(3x + 4y)$
6. $(8x - 3y)(8x + 3y)$
7. $(7x + 5y)(7x - 5y)$
8. $(4x + 9y)(4x - 9y)$
9. $(a + 1)(a - 1)$
10. $(3x - y)(3x + y)$

Example 6. Factor $4a^2 - 25b^2$.

Solution: $4a^2$ and $25b^2$ are two perfect squares. To obtain the factors of the difference of two perfect squares, proceed as follows:

1. Take the square root of each term. This gives $2a$ and $5b$.

2. Write these two terms twice to form two binomials and place a plus sign between the terms of the first binomial and a minus sign between the terms of the second. This gives $2a + 5b$ and $2a - 5b$.

3. Mentally obtain the product of these factors. It is $4a^2 - 25b^2$.

$$\therefore 4a^2 - 25b^2 = (2a + 5b)(2a - 5b)$$

Example 7. Factor $16a^4 - 625b^4$.

Solution: Following the procedure of Ex. 1 gives the factors $4a^2 + 25b^2$ and $4a^2 - 25b^2$. The second of these is not a prime factor but can be again factored.

The factors of $4a^2 - 25b^2$ are $2a + 5b$ and $2a - 5b$.

Altogether there are three factors:

$$4a^2 + 25b^2,\ 2a + 5b, \text{ and } 2a - 5b$$

The process is indicated as follows:

$$\begin{aligned} 16a^4 - 625b^4 &= (4a^2 + 25b^2)(4a^2 - 25b^2) \\ &= (4a^2 + 25b^2)(2a + 5b)(2a - 5b) \end{aligned}$$

Problems

Factor each of the following and check by multiplying:

1. $x^2 - y^2$
2. $x^2 - 4y^2$
3. $x^2 - 1$
4. $4x^2 - 9y^2$
5. $49a^2 - 64b^2$
6. $25x^2 - y^2$
7. $36a^2 - 1$
8. $121x^2 - 81y^2$
9. $144a^2 - 169b^2$
10. $81x^2 - 49y^2$
11. $100a^2 - 49b^2$
12. $225x^2 - 289y^2$
13. $121x^2 - 4y^2$
14. $x^4 - y^4$
15. $a^4 - 1$
16. $a^4 - 16$
17. $4x^4 - 9$
18. $16x^4 - y^4$
19. $81a^4 - 16b^4$
20. $256x^4 - 81y^4$

73. Miscellaneous Types.—The following expressions contain all of the different cases of factoring which we have studied. Some expressions are a combination of two of the types considered. When factoring them, the following guide may be of help:

1. Examine the expression and remove any monomial factors that may be present.

2. Examine the expression in the parenthesis and factor it if possible.

3. Examine the factors which you now have. Are they prime factors? If not, continue factoring until all factors are prime factors.

Example 8. Factor $6ax^2 - 22axy + 20ay^2$.

Solution: First remove the common factor $2a$, and then factor the trinomial.

$$\begin{aligned} 6ax^2 - 22axy + 20ay^2 &= 2a(3x^2 - 11xy + 10y^2) \\ &= 2a(3x - 5y)(x - 2y) \end{aligned}$$

Example 9. Factor $75a^2c - 12b^2c$.

Solution:
$$\begin{aligned} 75a^2c - 12b^2c &= 3c(25a^2 - 4b^2) \\ &= 3c(5a + 2b)(5a - 2b) \end{aligned}$$

Problems

Factor each of the following:

1. $4x^2 - 32x + 48$
2. $3x^2 - 18x + 15$
3. $12a^2 - 36a + 15$
4. $10x^2 + 5xy - 5y^2$
5. $6ax + 4axy - 8axy^2$
6. $2x^2 - 2$
7. $3x^2 - 12y^2$
8. $4ax^2 - 4a$
9. $8bx^2 + 4bxy - 4by^2$
10. $36ax^2 - 129axy + 105ay^2$
11. $9a^2 - 30ab + 25b^2$
12. $8x^4 - 18$
13. $72a^2x - 2x$
14. $126a^2 - 70ab + 12b^2$
15. $60ay^2 + 28ay - 16a$
16. $45a^3 + 6a^2b - 24ab^2$
17. $162x^2 - 98y^2$
18. $48ax^4 - 3ay^4$
19. $5a^2x^2 - 5a^2$
20. $10y^2 - 50y - 60$
21. $3a^3x + 9a^2y + 12ay^2$
22. $200a^3 - 98ab^2$
23. $242x^3 - 8xy^2$
24. $104ax^2 + 194axy - 90ay^2$

74. Quadratic Equations.—A quadratic equation is an equation which contains the second power of the unknown quantity but no higher power. In this section we shall consider only such quadratics as can be solved by factoring.

Example 10. Solve the equation $x^2 - x - 30 = 0$.

Solution:
$$x^2 - x - 30 = 0$$

By factoring,

$$(x + 5)(x - 6) = 0$$

Since the product of $x + 5$ and $x - 6$ is zero, either of these two factors may be equal to zero. We, therefore, equate each of these factors to zero and solve for x.

If $x + 5 = 0$, $x = -5$ If $x - 6 = 0$, $x = 6$ $\therefore x = 6$ or -5 *Ans.*

Check:

If $x = 6$

$$\begin{aligned} x^2 - x - 30 &= 0 \\ (6)^2 - 6 - 30 &= 0 \\ 36 - 6 - 30 &= 0 \\ 36 - 36 &= 0 \\ 0 &= 0 \end{aligned}$$

If $x = -5$

$$\begin{aligned} x^2 - x - 30 &= 0 \\ (-5)^2 - (-5) - 30 &= 0 \\ 25 + 5 - 30 &= 0 \\ 30 - 30 &= 0 \\ 0 &= 0 \end{aligned}$$

Every quadratic equation has two roots. These roots may be equal or unequal.

Example 11. Solve the equation $3x^2 - 25x + 28 = 0$.

Solution:

$$3x^2 - 25x + 28 = 0$$
$$(3x - 4)(x - 7) = 0$$

If $x - 7 = 0$, $x = 7$ If $3x - 4 = 0$, $3x = 4$, $x = \frac{4}{3}$ $\therefore x = 7$ or $\frac{4}{3}$ *Ans.*

Example 12. Solve the equation $15x^2 + 6x + 10 = 66 - 5x$.

Solution: Bring all terms into the left-hand member of the equation, thus leaving zero for the right-hand member. Combine similar terms and solve.

$$15x^2 + 6x + 10 = 66 - 5x$$
$$15x^2 + 6x + 10 - 66 + 5x = 0$$
$$15x^2 + 11x - 56 = 0$$
$$(5x - 8)(3x + 7) = 0$$

If $5x - 8 = 0$, $5x = 8$, $x = \frac{8}{5}$ If $3x + 7 = 0$, $3x = -7$, $x = -\frac{7}{3}$ $\therefore x = \frac{8}{5}$ or $-\frac{7}{3}$ *Ans.*

Problems

Solve the following quadratic equations by the method of factoring, and check your answers:

1. $x^2 - 11x + 24 = 0$
2. $x^2 + 5x - 14 = 0$
3. $x^2 + x - 110 = 0$
4. $x^2 + 7x + 12 = 0$
5. $x^2 + 14x + 45 = 0$
6. $x^2 + 56 = 15x$
7. $x^2 + 5x = 6$
8. $10x^2 + 7x - 6 = 0$
9. $6x^2 - 31x + 40 = 0$
10. $40x^2 - 41x + 10 = 0$
11. $20x^2 + 11x - 3 = 0$
12. $56x^2 - 37x + 6 = 0$
13. $12x^2 + 13x - 55 = 0$
14. $18x^2 - 28 - 55x = 0$
15. $55x^2 + 119x = -36$
16. $15x^2 + 6 = 19x$
17. $15x^2 - 6 = x$
18. $40x^2 + 31x = 33$
19. $10x^2 - 27x + 10 = 5 - 18x^2$
20. $15x^2 - 27x = 10x + 8$
21. $12x^2 + 20x + 34 = 9 - 20x$
22. $4x^2 + 13x - 18 = 17 - 10x$
23. $56 - 5x^2 = 26x - 8x^2$
24. $2x^2 - 12x = 10 - 3x^2 + 11x$

75. Review Problems:

1. What is the value of $7\frac{2}{3} - 2\frac{1}{2} \times 1\frac{3}{5} \div 3\frac{3}{4} \div 1\frac{9}{16}$?
2. Find the value of $\frac{3}{5} \times \frac{2}{6} \times \frac{7}{8} \div \frac{14}{6} \div \frac{3}{16} \times \frac{5}{4}$.
3. Find the value of $\frac{3}{5} + \frac{3}{4} \times \frac{2}{9} \div \frac{3}{16} \times \frac{8}{12} + 8$.
4. Solve for x in the equation $18x - 324 = 10x + 648 - 14x$.
5. Solve for y in the equation $15y + 7 - 3y = 2y + 57$.

6. The difference between two numbers is 43, and their sum is 129. Find the numbers.

7. A tree is $2\frac{1}{2}$ times as high as a tent. The sum of their heights is 175 ft. How high is each?

Solve the following equations:

8. $\frac{2x}{3} + \frac{7x}{4} = \frac{29}{24}$

9. $\frac{1}{42} = \frac{1}{6} + \frac{1}{X} + \frac{1}{7}$

10. $\frac{1}{y} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$

11. $\frac{1}{2} = \frac{1}{6} + \frac{1}{12} + \frac{1}{R}$

12. Find the resistance of an electric bell which requires 0.105 volt to send a current of 0.0025 amp. through it.

13. How many volts are required to force 0.017 amp. through a resistance of 3.014 ohms?

14. A generator maintains a pressure of 492 volts across a 20-ohm resistance, a lamp of 22.4 ohms resistance, and a heater of 97.6 ohms resistance connected in series. How much current flows?

15. Suppose the 20-ohm resistance in Prob. 14 to be replaced by an unknown resistance. What is the value of this resistance if a current of 3 amp. flows?

16. Three resistances of 7, 14, and 21 ohms are connected in parallel. What is the conductance of the combination? What is the resistance?

17. Three resistances of 6, 8, and 10 ohms are connected in parallel. Find the value of the resistance which must be placed in series with this group to make the total resistance 9 ohms.

18. Three lamps of 175 ohms, 105 ohms, and 140 ohms are connected in parallel. How much resistance must be connected in parallel with the three lamps to reduce the combined resistance to $27\frac{3}{11}$ ohms?

19. Three resistances of 2.5, 7.5 ohms, and 15 ohms are in parallel. It is desired to reduce the total resistance to 1.5 ohms by using a fourth resistance. What is the value of this resistance?

20. Suppose 175 volts are connected across the parallel resistances in Prob. 16. How much current would flow through each resistance?

21. Three resistances are connected in parallel across a generator whose brush potential is 126 volts. One resistance measures 18 ohms and the other 21 ohms. If the generator delivers 23.7 amp., what is the value of the third resistance?

22. Three resistances of 7.5, 9, and 10.5 ohms are connected in parallel. If the total current flowing is 16.05 amp., how much current is flowing through each resistance?

23. Four resistances of 3, 4.5, 6, and 7.5 ohms are connected in parallel. If a total of 46.2 amp. flows, how much current is flowing through each resistance?

24. What part of the total current will flow through each of three resistances of 4, 6, and 7 ohms, respectively, connected in parallel?

25. Three resistances of 2, 8, and 11 ohms are in parallel. What part of the total current flowing passes through the 2-ohm resistance?

26. Three resistances of 4, 7, and 8 ohms are in parallel. What part of the total current flowing passes through the 7-ohm resistance?

27. Find the total resistance of 4 ohms in series with 3, 5, and 6 ohms connected in parallel.

28. Express 3,200 watts in kilowatts and horsepower.

29. Express 42,575 watts in kilowatts and horsepower.

30. A motor is drawing the equivalent of 7.4 hp. from a 110-volt line. What is the line current?

31. What is the resistance of a 40-watt, 110-volt lamp?

Perform the operations indicated in the following:

32. $(4x^4 - 2x^3 - 6x + 1) - (x^3 + x^2 - 3)$

33. $(ax^2 - bx + c)(a + b)$

34. $3y^2 - (4x^2 - 3y^2) + 2x - (3x^2 + 4xy - 3y^2)$

35. $(3x^2y^2 - 4x + 5y^2)(8x^3y - 4xy^3 + y^5)$

36. $(125a^3b^3 + 25ab^4 - 5a^2b^5) \div (-5ab^3)$

37. $(6a^2 - 15b^2 - ab) \div (2a + 3b)$

38. $(-4x^4 - 22x^2 + 35x^6 + 5) \div (5x^4 + 3x^2 - 1)$

39. Subtract $4x^2 - 3x - 1$ from $3x^2 - 4x + 7$

40. Divide $x^3 - y^3$ by $x - y$

41. Solve the equation $4(x - 6) = 3x - 10$

42. Solve the equation $3(x - 2) + 15 = 5x - 3$

43. A, B, and C are partners. A contributes $750 more than C and B contributes a third as much as C. How much does each contribute if their total capital is $5,650?

44. A invests a third as much as B, and B invests twice as much as C. Together they invested $11,000. How much money did each invest?

45. $C = \frac{5}{9}(F - 32)$. Find F when $C = 56$.

46. $L' = L(1 + kt)$. Find k when $L' = 327$, $L = 326.94$, $t = 10$.

47. What is the diameter in mils of a wire whose area is 119,025 cir. mils?

48. What is the diameter of a copper wire 2,000 ft. long whose resistance is 0.4 ohm?

49. What is the resistance of a german silver (30 per cent nickel) wire 1,100 ft. long and 0.321 in. in diameter?

50. How many feet of copper wire will have the same resistance as the wire used in Prob. 49, the diameter of the copper wire being 0.321 in.?

51. How many feet of steel wire $\frac{1}{8}$ in. in diameter will have a resistance of 1 ohm?

52. How many amperes can two copper line wires 0.265 in. in diameter transmit over a distance of 600 ft. with a drop of 4 volts?

53. A group of lamps is 350 ft. from a generator. What size copper wire must be used between the generator and the lamps if the generator voltage is 114 and the voltage at the lamps should be 112, if the generator supplies 50 amp?

54. What is a right angle; an acute angle; an obtuse angle?

55. What is a triangle; a right triangle?

56. How many degrees are there in the three angles of any triangle?

57. The first angle of a triangle is 27° larger than the third and 12° smaller than the second angle. Find the three angles.

58. The second angle of a triangle is 12° larger than the first and 36° smaller than the third angle. Find the angles of the triangle.

59. Find the area of a triangle whose base is 127 ft. 8 in. and whose altitude is 39 ft. 9 in.

60. Find the area of a triangle whose base is 12 ft. and whose altitude is 26 ft.

61. Find the area of a trapezoid whose parallel sides measure 127 ft. and 155 ft. and whose altitude is 39 ft.

62. Find the area of a circle 38 ft. in diameter.

63. Find the area of a circle 58 in. in diameter.

64. Find the volume of a cylinder 6 ft. in diameter and 27 ft. long.

65. A copper busbar is 4 in. wide, $\frac{7}{16}$ in. thick, and 12 ft. long. How much does it weigh if copper weighs 0.3184 lb. per cu. in.?

66. How many circular mils are there in the busbar in Prob. 65?

67. A wire ribbon is 2.5 in. wide and $\frac{1}{8}$ in. thick. Find its area in circular mils.

68. A square bar measures 0.5 in. on a side. What is its circular-mil area?

Solve the following equations:

69. $\dfrac{3x}{5} + \dfrac{3}{8} - \dfrac{2x}{5} = \dfrac{71}{40}$

70. $\dfrac{5x}{8} + \dfrac{3}{7} - 7x = \dfrac{1}{8} - \dfrac{17}{28}$

71. $5x + \dfrac{4x - 6}{7} - 1$

72. $\dfrac{9x + 4}{3} = \dfrac{2x + 11}{5}$

73. $5x - \dfrac{7x + 3}{2} = \dfrac{2x + 8}{3}$

74. $2x - \dfrac{5x - 2}{7} = \dfrac{3x + 7}{2} + 7 - x$

75. $11x - \dfrac{3x + 5}{4} = \dfrac{21x - 1}{3} - \dfrac{2x - 9}{3}$

76. $\dfrac{2x - 8}{3} = \dfrac{x - 5}{2}$

77. $6x - \frac{8x - 6}{5} = \frac{8}{3}$

78. $4 - \frac{6x + 8}{5} = 14 + \frac{4x - 12}{3}$

79. $10x - \frac{4x + 26}{3} = 124 - \frac{14x - 102}{4}$

80. Factor $12x^3y + 20x^2y - 4xy$.

81. Factor $4x^2 - 9y^2$.

82. Factor $36a^2 - 1$.

83. Factor $2x^2 - 2$.

84. Find the diagonal of a square 120 ft. on a side.

85. Find the diagonal of a rectangle which is 370 ft. long and 248 ft. wide.

86. Find the width of a rectangle whose diagonal measures 1,521 ft. and which is 1,296 ft. long.

87. What is the maximum current-carrying capacity of a resistor marked "500 ohms, 200 watts"?

88. What should be the power rating of a resistor of 20,000 ohms which is to be connected across 400 volts?

89. What should be the maximum current carried by a 25,000-ohm resistor which can safely dissipate 10 watts? What is the maximum voltage across which this resistor can be connected?

90. A self-biasing 1,000-ohm resistor is rated at 1 watt. What is the maximum safe voltage drop for this resistor?

91. A resistor which can dissipate 25 watts has a resistance of 150 ohms. How many amperes can it carry? How many milliamperes?

92. A 625-ohm resistor is rated at 2 watts. How many milliamperes can be passed through it with safety? How many amperes?

CHAPTER XIV

THE SLIDE RULE

76. Description of the Slide Rule.—The slide rule, as usually constructed, consists of mahogany faced with celluloid and is about 10 in. long, 1½ in. broad, and ¼ to ⅜ in. thick. On the surface of the rule there are engraved several series of graduations, which form the various scales of the rule.

On the slide rule illustrated in Fig. 33, scales A and D are fixed scales, while scales B and C are placed on a slide between the two fixed scales. This slide is accurately fitted to a groove so that it may be easily moved from left to right and from right to left. The rule is also provided with a movable glass runner through the center of which there is engraved a fine vertical line. By means of this runner, coinciding points on scales A and D can readily be determined and it also permits fixing on the scale the intermediate results of a series of operations, thus obtaining greater accuracy in the final result.

The figures marked on the scale are arbitrary. For example, the figure 2 may represent the number 2 or any decimal part of 2 as 0.2, 0.02, etc. It may also represent 2 multiplied by any multiple of 10 such as 20, 200, 2,000, etc. The same is true of any other number marked on the scale or of any number represented by the smaller graduations.

The scales on the rule are logarithmic scales, that is, the position of any number on the scale is determined by the logarithm of that number. Adding the logarithms of two numbers gives the logarithm of the product of these numbers. By means of the slide on the slide rule, we can mechanically add two distances which represent the logarithms of any two numbers and read on the rule the number which corresponds to the sum of these two logarithms. The number read from the rule is the product of the two numbers whose logarithms

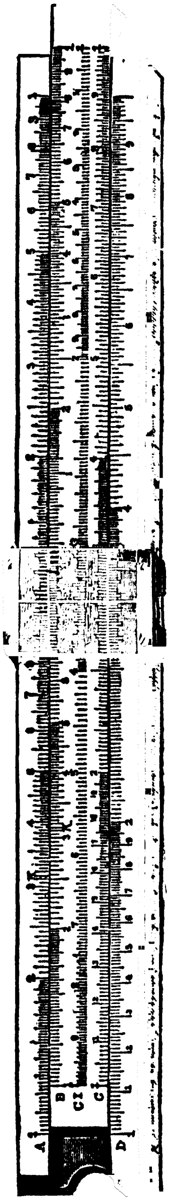


FIG. 33.—The slide rule.

were added on the slide rule. Similarly, by subtracting certain distances on the rule, we can perform division. There are also many other operations, such as square root and the solution of special types of formulas, which can be readily performed with the slide rule.

77. Operations with the Slide Rule.—The initial graduation at the left of any scale, marked "1," is termed the "left index" of the scale. The last graduation, on the opposite end of the scale, which is also marked "1," is termed the "right index." Multiplication and division are performed by using scales C and D. These operations are explained by the illustrative examples which follow.

Multiplication

Example 1. Multiply 11.17 by 6.

Solution: Set the left index of scale C on 11.17 (scale D); under 6 (scale C) read 67.

Figure 33 shows the slide rule set to read the answer to this example.

Example 2. Multiply 2.4 by 6.

Solution: Set the right index of scale C on 2.4 (scale D); under 6 (scale C) read 14.4.

It was necessary to set the right index of scale C on 2.4, because if the left index had been used, 6 would have fallen beyond the end of scale D.

Division

Example 3. Divide 247 by 19.

Solution: Set 19 (scale C) on 247 (scale D); under the left index of scale C read 13.

Example 4. Divide 2,660 by 28.

Solution: Set 28 (scale C) on 266 (scale D); under 1 (scale C) read 95.

Continued Multiplication and Division

Example 5. Multiply $13 \times 10.6 \times 8 \times 1.7$.

Solution: Set 1 (scale *C*) on 13, move runner to 10.6 (scale *C*), set 1 (scale *C*, right index) to runner, move runner to 8 (scale *C*), set 1 (scale *C*) to runner; under 1.7 read 1,874.

The position of the decimal point is best determined by a rough approximation of the result. To illustrate, 13(10.6) is roughly 130, 130(8) is roughly 1,000, 1,000(1.7) is roughly 1,700. For the purpose of fixing the decimal point, it is merely necessary to know that the answer is between 1,000 and 10,000.

Example 6. Find the value of

$$\frac{15 \times 74 \times 8.7}{38 \times 5.6 \times 28}$$

Solution: Set 38 (scale *C*) on 15 (scale *D*), move runner to 74 (scale *C*), set 5.6 (scale *C*) to runner, move runner to 8.7 (scale *C*) set 28 (scale *C*) to runner; under 1 (scale *C*) read 1.62.

The position of the decimal point is again determined by roughly estimating the result. $15 \div 38 = 0.5$, $0.5(74) = 37$, $37 \div 5.6 = 6$, $6(8.7) = 54$, $54 \div 28 = 2$. The decimal point is placed between the 1 and the 6, since the answer is estimated to be between 1 and 10.

Example 7. Find the value of

$$\frac{28 \times 9.4 \times 550}{15 \times 8.7 \times 113}$$

Solution: Set 15 (scale *C*) on 28 (scale *D*). The next operation is to move runner to 9.4 (scale *C*), which falls beyond the end of scale *D*. We first, therefore, move runner to 1 (scale *C*), set right index (scale *C*) to runner, and then move runner to 9.4. Set 8.7 (scale *C*) to runner, move runner to 550 (scale *C*), set 113 (scale *C*) to runner; under 1 (scale *C*) read 9.82.

To fix decimal point: $28 \div 15 = 2$, $2(9.4) = 18$, $18 \div 8.7 = 2$, $2(550) = 1{,}100$, $1{,}100 \div 113 = 10$. This estimate shows that the result is near 10. The decimal point is, therefore, placed between the 9 and the 8.

Square Root and Squares

Example 8. Find the square root of 169.

Solution: Set runner to 169 on left half of scale *A*; under runner on scale *D* read 13.

To take the square root of a number having an odd number of digits, use the left half of scale *A*; for a number having an even number of digits, use the right half of scale *A*.

Example 9. What is the value of $(25)^2$?

Solution: Set runner to 25 (scale *D*); on scale *A* read 625.

The square of a number of 2 digits contains either 3 or 4 digits. Since 625 was read from the left half of scale *A*, we know that the answer is a

number of 3 digits, *for if the answer had contained 4 digits it would have been read from the right half of scale A.*

Example 10. Find the value of $5\sqrt{196}$.

Solution: Set runner to 196 (left half of scale A), set index of scale C to runner; under 5 read 70.

Problems

Use a slide rule to determine the result of the operations indicated in the following group of problems:

1. 7×42

2. 9×18

3. 12×27

4. $492 \div 4$

5. $324 \div 6$

6. $981 \div 9$

7. 7×1.65

8. 12×1.39

9. 2.9×8.7

10. $765 \div 2.4$

11. $92.9 \div 4.7$

12. $6.2 \times 8 \times 9.12$

13. $5.4 \times 7 \times 19.6$

14. $9.16 \times 4.2 \times 13.5$

15. $\sqrt{289}$

16. $\sqrt{3.61}$

17. $\sqrt{0.0576}$

18. $(18)^2$

19. $(65)^2$

20. $(52)^2$

21. $\dfrac{23 \times 1.5 \times 10.9}{18 \times 2.7 \times 23}$

22. $\dfrac{47 \times 98 \times 225}{96 \times 39}$

23. $\dfrac{5.6 \times 3.4 \times 8.5}{4.7 \times 10.3}$

24. $\dfrac{\sqrt{925} \times 67 \times 13.6}{8.4 \times 95}$

25. $\dfrac{5.2 \times \sqrt{1581} \times 7.6}{92 \times 0.65}$

26.[1] $\dfrac{0.027 \times 0.043}{0.0065 \times 0.029}$

27. $\dfrac{1.39 \times \sqrt{3,825} \times 0.065}{0.092 \times 3.67}$

28. $\dfrac{36 \times 96.8 \times 25.9}{1.84 \times 3.6 \times 0.085}$

29. $\dfrac{65.2 \times 18.27 \times 31}{14.5 \times 0.94}$

30. $\dfrac{3.28 \times \sqrt{1,590} \times 1.9}{9.3 \times (26)^2}$

[1] NOTE.—In problems of this kind, it is usually convenient to eliminate the decimal points from numerator and denominator. This is done by moving the decimal points of a number in the numerator and a number in the denominator to the right, the same number of places for each number. For example, in Prob. 26, the decimal points of 0.027 and 0.0065 can each be moved four places to the right so that the numbers become 270 and 65. Also, 0.043 and 0.029 are changed to 43 and 29.

CHAPTER XV

GRAPHS

78. Use of Graphs.—The graph is a convenient method for representing statistics, the results of tests, etc. Its use enables the presenting of facts in such a way that a picture of the conditions is obtained and the trend of the changes illustrated by the graph becomes at once apparent. Squared paper, described in Sec. 34, is usually used to present statistics in graphical form, by means of a curved-line or broken-line graph.

Circle graphs and bar graphs are more convenient than the line graph when it is desired to represent the relative size of different quantities at a given time, such as the populations of a number of cities in 1920. All graphs in this chapter are to be line graphs.

79. Curved and Broken-line Graphs.—When plotting a graph which represents the results of a laboratory experiment, a smooth curved line should be drawn through the average path represented by the points on the graph paper. This is true because the conditions under which the experiment is performed are subject to change during the course of the experiment and because of errors in the reading of instruments, etc., which make it impossible to determine the location of each point with absolute accuracy. Such a graph is called a *curved-line graph.* If the points of the graph representing the results of an experiment could be accurately determined, the curved-line graph would still be used, because the changes which it represents occur gradually. If we were to plot a graph showing the daily sales on the New York Stock Exchange, we should use a *broken-line graph,* because the total sales from day to day vary abruptly. A broken-line graph is constructed by connecting each point of the graph with the preceding point by means of a straight line.

80. Construction of Graphs.—To construct a graph, begin by choosing a point near the left-hand corner of the graph paper. This is called the point of "origin," and from it we lay off the vertical and horizontal scales. Study the figures which represent the magnitudes which are to be plotted, take note of the largest and the smallest value, and then choose a scale which will cause these values to fall within the limits of the graph paper. When the two scales have been fixed, plot each point of the curve, and when these have all been determined, draw the curve which will best fit the conditions which are to be pictured.

Example 1. Plot a curve showing the relation between the rated output and the weight of turbo-alternator sets from the data given in the following table:

Frequency, cycles per second	Rated output, kilovolt-amperes	Weight of set, pounds
25	2,500	122,600
25	5,000	216,400
25	12,500	457,300
25	25,000	792,000

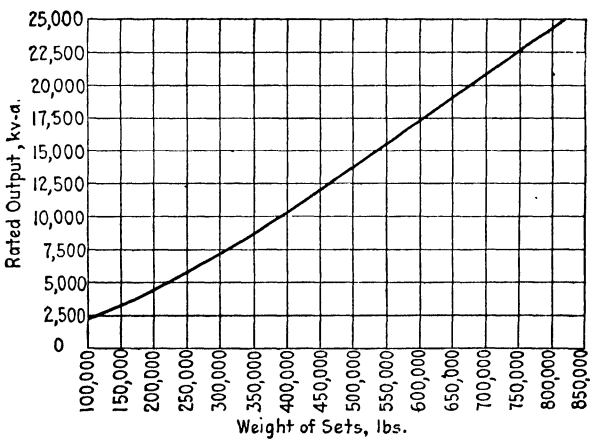


FIG. 34.

Solution: From the origin, mark off to the right equal spaces to represent the weights of the sets. The weights given lie between 122,000 and 792,000, a range of 670,000 lb. There are about 15 squares to the right of the origin which we can conveniently use. If each of these squares represents a difference of 50,000 lb., the total change in weight can be represented by 14 squares. Let the origin represent 100,000 lb. and each square to the right an additional 50,000 lb.

For the vertical scale, which represents the rated output of the generators, begin with zero at the origin and let each square represent 2,500 kva.

We can now locate the points of the graph. On the horizontal scale, locate 122,600 and move vertically upward until the horizontal line which represents 2,500 kva. is reached. Mark the intersections as the first point of the graph. In the same way, determine the other points. Figure 34 shows the completed graph, which is a curved-line graph.

From Fig. 34, determine the weight of a turbo-alternator set whose rated output is 17,500 kva.

Problems

From the data given, plot either curved- or broken-line graphs, whichever better illustrates the conditions of the problem. The data for Prob. 1 to 8 were obtained from the "Standard Handbook for Electrical Engineers."

1. The change in the specific gravity of the solution in an automobile battery with decrease of voltage during a starting test is given by the table below:

E.m.f. per cell	Specific gravity
2.00	1,300
1.99	1,295
1.96	1,280
1.93	1,280
1.88	1,275
1.84	1,270
1.80	1,265
1.72	1,255
1.52	1,250

2. Table of sulphuric-acid solutions

Specific gravity of solution, 70° F	Percentage of sulphuric acid in solution
1 11	16 0
1 12	17 4
1 14	20 1
1 16	22 7
1 17	24 0
1 19	26 5
1 21	29 0
1 23	31 4
1 25	33 7
1 26	35 0
1 29	38 5
1 32	42 0
1 36	46 3
1 40	50 5
1 50	60 15
1 60	69 12
1 70	77 6
1 80	87 5

3. Efficiencies of standard transformers, 60 cycles

Rating, kilovolt-amperes	Full load efficiency, per cent
1 0	95 8
1 5	96 2
2 0	96 5
3 0	96 8
5 0	97 3
7 5	97 6
10 0	97 8
15 0	97 9
20 0	98 0
25 0	98 2
30 0	98 2
40 0	98 3
50 0	98 4

4. Weight of lead-covered paper-insulated cable, 19 A.W.G.:

Number of pairs	Approximate weight per foot, pounds
5	0.640
10	0.850
15	0.970
20	1.138
25	1.264
30	1.390
40	1.643
50	1.995
60	2.220
75	2.584
100	3.738
120	4.221
150	4.865
200	5.808
300	7.587

5. Maximum permissible length of line for closed-circuit Morse (duplex, two sides):

Resistance per mile, ohms	Length of line, miles
2	783
3	658
4	580
6	485
8	425
10	384
15	318
20	278
25	250
30	229
40	200
50	180

6. Sag table for hard-drawn bare copper line wire, 12 S.W.G., 150 ft. span:

Temperature, degrees Fahrenheit	Sag, inches
−30	4.5
−10	5.0
10	6.0
30	7.0
60	9.0
80	11.5
100	14.0

7. Calculated relation between antenna current and distance for two ships with antenna heights 130 ft. and wave length 1,000 m.

Antenna current, amperes	Working distance, miles
1	75
2	135
3	180
5	235
7	280
10	345
15	420
20	475
25	525
30	565
40	630
50	685
60	725

8. Test data on vertical windmill, 12-ft. aermotor (steel), 2-lb. load on prony brake:

Wind velocity, miles	Output, horsepower
8	0.089
12	0.285
16	0.386
20	0.458
25	0.523

9. Building-construction in New York City:

Year	Number of buildings
1917	8,447
1918	6,322
1919	25,402
1920	19,436
1921	37,105
1922	51,750
1923	70,994
1924	60,479
1925	61,501
1926	58,047
1927	47,071

10. Production of silver in the United States:

Year	Value of silver, millions of dollars
1917	59 08
1918	66 49
1919	63 53
1920	60 80
1921	53 05
1922	56 24
1923	60 13
1924	43 82
1925	45 91
1926	39.14
1927	34 27

11. Precipitation at New York City during 1927:

Month	Precipitation, inches
January	1 95
February	3 33
March	1 18
April	2 66
May	3 67
June	3 13
July.	5 93
August	8 05
September	3 84
October	8 82
November	3 95
December ...	3 39

12. Commercial failures in the United States:

Year	Number of failures
1913	16,037
1914	18,280
1915	22,156
1916	16,993
1917	13,855
1918	9,982
1919	6,451
1920	8,881
1921	19,652
1922	23,676
1923	18,718
1924	20,615
1925	21,214
1926	21,773
1927	23,146

CHAPTER XVI

SOLUTION OF PROBLEMS INVOLVING PERCENTAGE

81. Percentage.—A knowledge of percentage is very useful and necessary in all walks of life. We shall consider the solution of a problem involving percentage as a problem in simple equations.

Per cent means "by the hundred." Five per cent of any quantity, therefore, is 0.05 of the quantity.

Change the following to decimal fractions:

5%, 10%, 12½%, 15%, 20%, 45%, 63%, 87%, 1%, ½%, ¼%.

82. Percentage Formula.—In percentage problems we deal with three quantities, a certain original quantity, a rate of interest on that quantity or a certain per cent of the quantity, and the part of the original quantity determined by the rate.

The original quantity is called the *base.*

The rate of interest or the number of per cent taken is called the *rate.*

The sum determined by the application of the rate to the base is called the *percentage.*

These three quantities always bear the same relation to each other as is expressed by the formula

$$P = BR$$

where P = percentage
B = base
R = rate

If you will substitute in this formula each time you have a percentage problem to solve, you will find them all very simple.

Example 1. How much is 20 per cent of 60?

Solution: $P = BR,\ B = 60,\ R = 0.20$

$$\therefore P = 60(0.20)$$
$$P = 12 \textit{ Ans.}$$

Example 2. What per cent of 90 is 18?
Solution: $P = BR$, $P = 18$, $B = 90$

$$\therefore 18 = 90R$$
$$90R = 18$$
$$R = 0.20 = 20 \text{ per cent } \textit{Ans.}$$

Problems

1. How much is 10 per cent of 12; of 15?
2. How much is 25 per cent of 38; of 44; of 60?
3. How much is 2 per cent of 25; 3 per cent of 15; ½ per cent of 50?
4. What per cent of 20 is 10?
5. What per cent of 40 is 5?
6. What per cent of 450 is 90?
7. What per cent of 80 is 8?
8. What per cent is 72 of 240?
9. 30 is 10 per cent of what number?
10. $2.00 is 5 per cent of how many dollars?
11. The interest on a certain sum of money at 6 per cent is $120.00. What is the sum of money?
12. The input to an electric motor is 7 kw. If the motor delivers 80 per cent of this power, how many horsepower does it deliver?
13. The input to an electric motor is 7 kw. If it delivers 7.1 hp., what per cent of the input is delivered?
14. A boy has 10 cts. This is only 40 per cent of the cost of the article which he wishes to buy. What is the cost of this article?
15. An article is listed at $15.00 with discounts of 30 per cent, 10 per cent, and 3 per cent for cash. What is the net cost?
16. An article is listed at $26.00 with discounts of 25 per cent, 12 per cent, and 2 per cent for cash. Find the net cost.

Example 3. A mechanic received an increase of 15 per cent in his wages, which raised his daily wage to $10.35. What was his daily wage before the increase?
Solution: Let x = mechanic's wage before the increase
Then

$$0.15x = \text{amount of the increase}$$
$$x + 0.15x = \text{mechanic's wage after the increase}$$
$$\therefore x + 0.15x = 10.35$$
$$1.15x = 10.35$$
$$x = \$9.00 \ \textit{Ans.}$$

17. A dealer sells an electric fan for $26.50, thereby gaining 12 per cent of the cost. What was the cost?
18. A man received an increase of 12½ per cent in his wages. He then received $5.15 per day. How much per day was he receiving before the increase?

19. A forging weighed 125 lb. before finishing and 111 lb. after finishing. What per cent of the original was turned off?

83. Efficiency.—No machine is 100 per cent efficient, by which we mean that no machine delivers so much power as is put into it. The efficiency of any machine is found by dividing the output of the machine by the input. Expressed as an equation, this is:

$$\text{Efficiency} = \frac{\text{output}}{\text{input}}$$

Example 4. What is the efficiency of a motor which draws 4.5 kw. from a line and delivers 5 hp.?

Solution:

$$4.5 \text{ kw.} = 4{,}500 \text{ watts}$$
$$5 \text{ hp.} = 5(746) \text{ watts} = 3{,}730 \text{ watts}$$
$$\text{Efficiency} = \frac{\text{output}}{\text{input}}$$
$$\text{Efficiency} = \frac{3{,}730}{4{,}500}$$
$$= 0.829 = 82.9 \text{ per cent } \textit{Ans.}$$

Example 5. How many kilowatts are being used by a motor which is delivering 6.7 hp. if its efficiency is 80 per cent?

Solution:

Motor output $= 6.7$ hp. $= 4{,}998$ watts

$$\text{Efficiency} = \frac{\text{output}}{\text{input}}$$
$$\therefore 0.80 = \frac{4{,}998}{x}$$

Multiplying both members by x,

$$0.80x = 4{,}998$$
$$x = 6{,}247.5 \text{ watts}$$
$$x = 6.25 \text{ kw. } \textit{Ans.}$$

Problems

1. Find the efficiency of a motor which draws 7 kw. from the line and delivers 8.5 hp.

2. Find the efficiency of a motor which delivers 14 hp. and draws 12.5 kw. from the line.

3. A motor delivers 31.5 hp. and draws 27 kw. from the line. What is its efficiency?

4. A turbine delivers 15 hp. to a generator whose output is 40 amp. at 225 volts. What is the efficiency of the generator?

5. The input to a generator is 50 hp. and the generator delivers 75 amp. at 440 volts. What is its efficiency?

6. A generator whose e.m.f. is 1,100 volts delivers 38.6 amp. What is the efficiency of the generator if its input is 62 hp.?

7. A motor has an efficiency of 83 per cent. If it draws 25 amp. from a 110-volt line, how many horsepower does it deliver?

8. How many kilowatts are used by a motor which is 85 per cent efficient and delivers 13 hp.?

9. The input to a generator is 5.11 hp. If the generator efficiency is 81 per cent, what is its output in kilowatts?

10. How many horsepower are necessary to drive a generator whose output is 33.1 amp. at 220 volts, if its efficiency at this load is 87 per cent?

11. How many kilowatts are used by a motor which is 82 per cent efficient when delivering 7 hp.?

12. How much current will a motor draw from a 220-volt line when delivering 9 hp., if its efficiency at this load is 87 per cent?

13. A motor draws 22 amp. from a 220-volt line. What is its horsepower output if its efficiency is 75 per cent?

14. How many kilowatts are being used by a motor whose efficiency is 90 per cent when delivering 25 hp.?

15. A generator delivers 20 amp. at 440 volts. How many horsepower are necessary to drive it, if its efficiency is 86 per cent?

16. How much current will a motor draw from a 220-volt line when delivering 13.5 hp., if its efficiency at this load is 89 per cent?

17. The input to a generator is 14 hp. and its efficiency is 79 per cent. What is the generator voltage if it delivers 75 amp.?

18. A motor draws 18 amp. from a line when delivering 10.5 hp. What is the line voltage if the motor is 90 per cent efficient at that load?

19. A turbine delivers 31 hp. to a generator whose load is 46.2 amps. What is the e.m.f. of the generator which is 88 per cent efficient at this load?

20. A motor whose efficiency is 85 per cent is taking 28 amp. from a line while delivering 17.5 hp. What is the line voltage?

21. A motor draws 25 amp. from a 115-volt line. What is its horsepower output if its efficiency at this load is 75 per cent?

22. Find the horsepower output of a motor whose efficiency is 82 per cent while drawing 45 amp. from a 220-volt line.

23. Find the cost of operating a 20-hp. motor for 200 hr. if the average load is 85 per cent of full load and the average motor efficiency is 80 per cent. The cost of power is $3\frac{1}{2}$ cts. per kilowatt-hour.

24. Find the cost of operating a 15-hp. motor 15 hr. a day for 330 days. The average load is 90 per cent of full load and the average motor efficiency is 82 per cent. The cost of power is 3 cts. per kilowatt-hour.

25. Find the cost of operating a 35-hp. motor for 30 days, 24 hr. each day, if the average load on the motor is 80 per cent of the full load and the average efficiency is 75 per cent. The cost of power is $2\frac{1}{2}$ cts. per kilowatt-hour.

26. What must be the horsepower output of a steam engine which is driving a 15-kw. generator. The generator is delivering a 15 per cent overload and its efficiency is 90 per cent.

27. What is the horsepower output of a turbine which is driving a 100-kw. generator. The generator is delivering a 15 per cent overload and its efficiency is 92 per cent.

Example 6. A turbine delivers 60 hp. to a generator whose terminal e.m.f. is 550 volts. What is the line current, if the generator efficiency is 85 per cent? What is the cost per kilowatt-hour of supplying this electrical energy if the cost of operating the turbine is 0.9 ct. per horsepower-hour of output?

Solution: 1. Find the generator output.

$$\text{Generator input} = 60 \text{ hp.} = 60(746) \text{ watts} = 44{,}760 \text{ watts}$$

Let

$$x = \text{generator output}$$

Then

$$\begin{aligned} 0.85 &= \frac{x}{44{,}760} \\ 44{,}760(0.85) &= x \\ x &= 38{,}046 \text{ watts generator output} \end{aligned}$$

2. Find the line current.

$$\begin{aligned} W &= EI \\ 38{,}046 &= 550(I) \\ 550I &= 38{,}046 \\ I &= 69.2 \text{ amp. line current} \end{aligned}$$

3. Find the cost of supplying the electrical energy.

$0.009(60) = $0.54, cost of operating generator for 1 hr. Generator output, each hour, is 38.046 kw-hr.

$$\text{Therefore, the cost per kilowatt-hour} = \frac{0.54}{38.046}$$

$$= \$0.014 \text{ per kilowatt-hour.}$$

28. A turbine is delivering 55.5 hp. to a generator which is supplying current at 550 volts. The efficiency of the generator is 88.5 per cent. What is the line current? The cost of supplying energy to the generator is 0.9 ct. per horsepower-hour. What is the cost per kilowatt-hour of producing the electrical energy?

29. The input to a generator whose terminal voltage is 220 volts is 107.5 hp. If the generator efficiency is 90 per cent, what is the line current? What is the cost per kilowatt-hour of producing this electrical energy, if the cost of supplying energy to the generator is 0.85 ct. per horsepower-hour?

30. A 11,000-volt generator has a full-load efficiency of 91 per cent and its input at full-load is 535 hp. What is the current at full load?

The cost of driving this generator is 0.9 ct. per horsepower-hour. What is the cost per kilowatt-hour of producing the electrical energy?

Example 7. A 300-hp. motor is connected to a generator by two line wires whose total resistance is 2.5 ohms. The generator output is 264 kw. and the line current when the motor is operating at full load is 24 amp. Find the efficiency of the motor.

Solution: 1. Find the power lost in the line.

$$\begin{aligned} W &= I^2R \\ W &= (24)^2(2.5) \\ &= 576(2.5) \\ &= 1{,}440 \text{ watts lost in the line} \end{aligned}$$

2. Find the input to the motor.

The total power supplied by the generator is 264,000 watts. Of this total, 1,440 watts are lost in the line, and the remainder is the power used by the motor.

$$\therefore 264{,}000 - 1{,}440 = 262{,}560 \text{ watts motor input}$$

3. Find the motor efficiency.

$$\text{The motor output} = 300 \text{ hp.} = 223{,}800 \text{ watts}$$

$$\text{Efficiency} = \frac{223{,}800}{262{,}560}$$

$$= 0.852 = 85.2 \text{ per cent.}$$

31. A generator supplies power to a motor whose output is 18 hp. over a line whose total resistance is 0.25 ohm. The generator output is 17 kw. and the current is 80 amp. What is the motor efficiency?

32. A motor which is delivering 83.5 hp. is drawing 40 amp. from a line whose total resistance is 0.3 ohm. The total power input to the line is 88 kw. Find the efficiency of the motor.

33. A two-wire generator is supplying 110 amp. to a 30-hp. motor operating at full load. The efficiency of the motor is 90 per cent and the resistance of each line wire between generator and motor is 0.095 ohm. What is the terminal voltage of the generator?

34. A motor is drawing 70 amp. from a generator over a line whose total resistance is 0.3 ohm. The motor delivers 35 hp. and its efficiency is 87 per cent. What is the generator e.m.f.?

35. A generator is supplying 36 amp. to a 10-hp. motor operating at full load. The efficiency of the motor is 89 per cent and the resistance of each line wire connecting generator and motor is 0.2 ohm. What is the brush potential of the generator?

36. In Prob. 35, suppose the generator supplies 50 amp. to a 15-hp. motor 86 per cent efficient, line resistance as before. What is the brush potential of the generator?

37. A generator supplies 43.56 kw. to a line whose resistance is 0.2 ohm. The power lost in the line is 0.5 per cent of the power supplied to the line. What is the line current?

38. What is the current in a line whose resistance is 0.18 ohm, if the power supplied to the line is 90.75 kw. and 0.6 per cent of this power is lost in the line?

39. A generator supplies 12 kw. to a line. The line resistance is 0.15 ohm and the power lost in the line is 0.5 per cent of the power supplied to the line. What is the line current?

40. A generator is supplying 15 amp. to a $1\frac{3}{4}$-hp. motor operating at full load. What is the line resistance? Motor efficiency is 80 per cent, e.m.f. at generator 113 volts.

41. A generator is supplying 20 amp. to a 5-hp. motor operating at full load. If the motor efficiency at this load is 85 per cent what is the line resistance, if the brush potential of the generator is 224 volts?

42. A motor delivers 25 hp. It is operated over a line whose total resistance is 0.22 ohm by a generator which supplies 21.56 kw. to the line. Line current is 47 amp. What is the efficiency of the motor at this load?

43. A motor delivers 20 hp. The line between motor and generator has a total resistance of 0.35. The line current is 76 amp. What is the efficiency of the motor if the generator output is 19.85 kw.?

44. A motor draws 7 kw. from a line. The line resistance between motor and generator is 0.42 ohm. If the power lost in the line is 0.6 per cent of the power supplied to the motor, what is the line current?

45. A motor delivers 5 hp. and is 84 per cent efficient. The resistance of the line between the generator and motor is 0.38 ohm. If the power lost in the line is 0.75 per cent of the power supplied to the motor, what is the line current?

CHAPTER XVII

PARALLEL CIRCUITS

84. Distribution System.—Most of the distribution systems in America are parallel systems; that is, a certain voltage is impressed across a pair of wires, called "feeders," and groups of lamps, motors, and other electric appliances are connected across the main feeders. This system of distribution produces a circuit which is a combination of series and parallel circuits.

We have learned that in a parallel circuit the current divides part of the total current flowing through each of the parallel paths. The total current is always the sum of the currents in the separate parallel branches. Suppose that we have 10 lamps connected in parallel and each lamp draws 0.5 amp. Evidently, the total current in the line is 5 amp.

Let us consider a simple parallel circuit and determine the voltage and currents in the circuit.

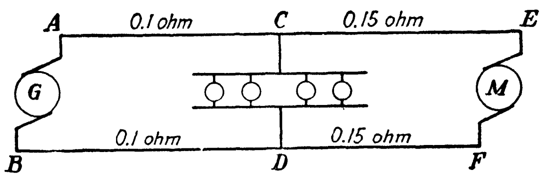


FIG. 35.

Example 1. Suppose we have a generator delivering current to a group of lamps and a motor, connected as shown in Fig. 35. The motor is drawing 7 amp. and each lamp draws 0.5 amp. The generator voltage is 112 and we wish to determine the voltage across the lamps and the voltage across the motor.

Solution: The problem resolves itself into this: We know the generator voltage. We also know that whenever a current passes through a resistance a certain voltage is required to cause the current to flow.

In this case, there is current flowing through wires AC and BD and a definite voltage will be used up or lost in forcing the current through these wires. The same holds true for the wires CE and DF. If, therefore, we can determine the voltage lost in the wires between the generator and the lamps, we can find the voltage at the lamps, and, similarly, we can find the voltage at the motor.

The problem, then, is to find the voltage lost in each section of the line. The loss is called the "line drop." The first step will be to determine the current flowing in each line wire, and then, knowing the resistance, we can calculate the line drop from Ohm's law $E = IR$.

In problems of this kind, always start with the current in the section farthest from the generator.

The motor draws 7 amp. Since there is no other path for the current to follow, all of it must flow through wires CE and DF.

Each lamp draws 0.5 amp. There are four lamps, so that the total current through the lamps from C to D is 2 amp.

The current for the motor and the current for the lamps is furnished by the generator. Since the motor takes 7 amp. and the lamps 2 amp., the current through the generator must be 9 amp. This 9 amp. flows away from the generator through wire AC and returns through wire BD. The current through these two wires is, therefore, the sum of the currents to the lamps and to the motor. At C the current divides, 2 amp. flow down through the lamps, and the remaining current toward the motor.

Now that we know the current in each section, we can find the voltage drop.

The total line resistance between AB and CD = 0.2 ohm. Current in these wires = 9 amp.

Voltage drop in these wires $9 \times 0.2 = 1.8$ volts
Voltage at CD = generator voltage − line drop
Voltage at $CD = 112 - 1.8 = 110.2$ volts

The next step is to determine the voltage lost in the second section of the line.

Total line resistance, lamps to motor = 0.3 ohm
Current through these wires = 7 amp.
Voltage drop CD to $EF = 7 \times 0.3 = 2.1$ volts
Voltage at motor = voltage at lamps − line drop
Voltage at motor $= 110.2 - 2.1 = 108.1$ volts

The important thing to remember in problems like the above is that the current in the line from the generator is not only the current to the lamps or the current to the motor but is the sum of the two.

Example 2. In Fig. 36, motor M_1 draws 5 amp., M_2 draws 7 amp., M_3 draws 4 amp., M_4 draws 9 amp., and M_5 draws 3 amp. Find the voltage drop in each section of the line and the voltage across each motor. The generator e.m.f. is 115 volts.

Solution: 1. Determine the line current in each section of the line.

The current through M_1 is 5 amp. The current in the line between M_1 and M_2 is, therefore, 5 amp.

The current through M_2 is 7 amp. and between the generator and M_2 the current is 12 amp., since the current for M_1 also passes through this section of the line.

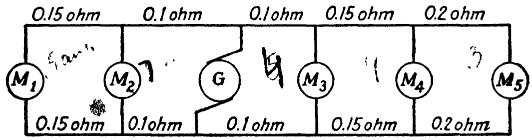


FIG. 36.

Similarly: The current between M_4 and M_5 is 3 amp.
The current between M_3 and M_4 is 12 amp
The current between G and M_3 is 16 amp.

2. Find the line drop in each section.

The total line resistance between M_1 and M_2 is 0.3 ohm. The line drop in the line is equal to the line resistance multiplied by the current in the line.

$\therefore$ Line drop M_1 to M_2 = 0.3(5) = 1.5 volts

Similarly:
Line drop G to M_2 = 0.2(12) = 2.4 volts
Line drop M_4 to M_5 = 0.4(3) = 1.2 volts
Line drop M_3 to M_4 = 0.3(12) = 3.6 volts
Line drop G to M_3 = 0.2(16) = 3.2 volts

3. Find the voltage at each motor.

The voltage at M_2 is the generator voltage less the line drop between G and M_2, the voltage at M_1 is the voltage at M_2 less the line drop between M_1 and M_2, etc. The generator e.m.f. is 115 volts.

$\therefore$ The e.m.f. at M_2 is 115 − 2.4 = 112.6 volts
The e.m.f. at M_1 is 112.6 − 1.5 = 111.1 volts
The e.m.f. at M_3 is 115 − 3.2 = 111.8 volts
The e.m.f. at M_4 is 111.8 − 3.6 = 108.2 volts
The e.m.f. at M_5 is 108.2 − 1.2 = 107.0 volts

Problems

1. A trolley system is a parallel system. Suppose there are three cars in one section of a trolley system, one car at A, one at B, and the

other at C. Car at A draws 15 amp., car at B 17 amp., and car at C 20 amp. What is the current from B to C; from A to B; from generator to point A?

2. Assume a distribution system, as is shown in Fig. 35. There are ten lamps, each drawing 0.7 amp. The motor requires 12 amp. What is the current in the line between the lamps and the motor? How much current in the line between the generator and the lamps? What is the line drop from CD to EF; from AB to CD?

3. Suppose that there are four trolley cars on a line, with generator at A and cars at points B, C, D, and E. Car at B draws 35 amp., car at C 30 amp., car at D 40 amp., and car at E 33 amp. How much current is furnished by the generator? How much current between D and E; between C and D; between B and C?

4. In Prob. 3, the total resistance of the line between points A and B is 1.5 ohms, between B and C it is 0.8 ohm, between C and D it is 1.2 ohms, and between D and E 0.6 ohm. Find the line drop in each section of the line.

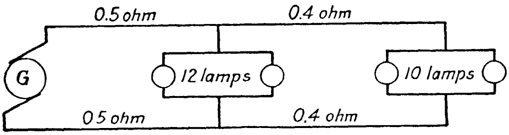


FIG. 37.

5. Each lamp in Fig. 37 takes 1 amp. The generator e.m.f. is 130 volts. Find the voltage across each group of lamps.

6. Draw a figure like Fig. 37 but put 8 lamps in the group nearest the generator and 11 lamps in the other group. The generator voltage is unknown. Each lamp in the first group draws 0.7 amp. and each lamp in the second group 0.5 amp. Find the generator voltage, if the e.m.f. across the first group is 112 volts.

7. In Fig. 37, replace the first group of lamps with a group of 20 lamps and the second group with one of 12 lamps. Line resistance as in Fig. 37. The e.m.f. across the first group is 110 volts and each lamp takes 0.6 amp. Find the generator voltage and the voltage across the second group of lamps.

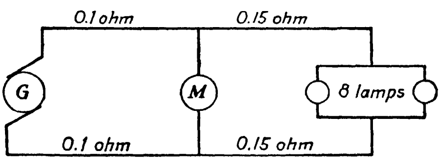


FIG. 38.

8. In Fig. 38, each lamp draws 0.85 amp. and the e.m.f. at the motor is 112 volts. The motor current is 15 amp. Find the voltage at the lamps and at the generator.

9. In Prob. 8, suppose the voltage at the motor to be 115 volts, generator e.m.f. 117 volts, and 10 lamps in the group each drawing 0.6 amp. What is the voltage at the lamps and what is the motor current?

10. Draw a diagram like Fig. 38. Each line wire between generator and motor has a resistance of 0.5 ohm, and each wire leading from the motor to the lamps has a resistance of 0.7 ohm. The motor draws 25 amp. and there are 30 lamps in the group each of which draws 1 amp. The generator e.m.f. is 120 volts. Find the voltage at the motor and at the lamps.

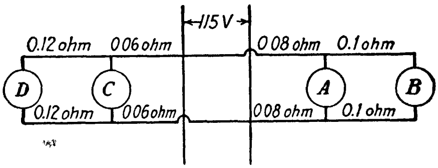


FIG. 39.

11. In Fig. 39, find the voltage at A, at B, at C, and at D when the currents are as follows:

$$A = 12 \text{ amp.}, B = 9 \text{ amp.}, C = 15 \text{ amp.}, D = 7 \text{ amp.}$$

12. Repeat Prob. 11 with the following currents: $A = 15$ amp., $B = 8$ amp., $C = 12$ amp., and $D = 10$ amp.

13. Repeat Prob. 11 with the currents as follows: $A = 20$ amp., $B = 22$ amp., $C = 30$ amp., and $D = 16$ amp.

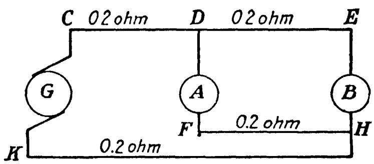


FIG. 40.

Example 3. In Fig. 40, A represents a group of 12 lamps and B a group of 16 lamps. The generator e.m.f. is 114 volts and each lamp takes 0.5 amp. Find the voltage at A and at B.

Solution: The current in wires CD and KH is 14 amp., the current in DE is 8 amp., and in FH it is 6 amp.

$$\begin{aligned}
\text{Voltage drop in } CD &= 0.2(14) = 2.8 \text{ volts} \\
\text{Voltage drop in } KH &= 0.2(14) = 2.8 \text{ volts} \\
\text{Voltage drop in } FH &= 0.2(6) = 1.2 \text{ volts}
\end{aligned}$$

The total voltage drop in these three wires is 6.8 volts. These three wires connect group A to the generator, and the voltage at A must be

equal to the generator voltage less the voltage drop in the wires leading from the generator to group A.

$$\therefore \text{The voltage at } A = 114 - 6.8 = 107.2 \text{ volts.}$$

In the same way, the voltage at B is found by subtracting from the generator e.m.f. the voltage drop in the wires leading from the generator to B, as follows:

$$\begin{aligned} \text{Voltage drop in } CD &= 0.2(14) = 2.8 \text{ volts} \\ \text{Voltage drop in } DE &= 0.2(8) \ \ = 1.6 \text{ volts} \\ \text{Voltage drop in } HK &= 0\ 2(14) = 2.8 \text{ volts} \end{aligned}$$

The total voltage drop in these wires is 7.2 volts.

$$\therefore \text{The voltage at } B = 114 - 7.2 = 106.8 \text{ volts.}$$

14. Draw a diagram like Fig. 40 and mark the line as follows: CD = 0.15 ohm, DE = 0.25 ohm, FH = 0.25 ohm, and KH = 0.3 ohm. If the generator e.m.f. is 115 volts and A represents a group of 50 lamps and B a group of 40 lamps, find the voltage at A and at B. Each lamp draws 0.7 amp.

15. Repeat Prob. 14 with the following values: CD = 0.12 ohm DE = 0.2 ohm, FH = 0.2 ohm, and KH = 0.25 ohm. The current through A is 30 amp., the current through B is 45 amp., and the generator e.m.f. is 118 volts.

16. Repeat Prob 14 with the following values: CD = 0.035 ohm, DE = 0.045 ohm, FH = 0.045 ohm, and KH = 0.035 ohm. A represents a group of 32 lamps each drawing 0.75 amp. and B is a motor drawing 12 amp. The generator e.m.f. is 120 volts.

17. Repeat Prob. 14 with the following values: CD = 0.038 ohm, DE = 0.042 ohm, FH = 0.042 ohm, and KH = 0.048 ohm. The current through A is 25 amp., through B it is 21 amp., and the generator e.m.f. is 120 volts.

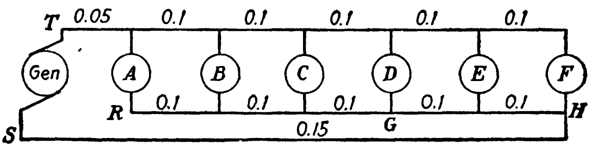


FIG. 41.

18. Figure 41 represents a distribution system and each of the circles represents a consumer. A uses 6 amp., B uses 8 amp., C uses 12 amp., D uses 10 amp., E uses 11 amp., and F uses 13 amp. Find the current in each section of the line.

19. With the line resistances as indicated in Fig. 41, a generator e.m.f. of 125 volts, and the currents as given in Prob. 18, find the voltage at each consumer.

20. Repeat Prob. 19, but disconnect wire SH from H and connect it directly to point G.

21. Repeat Prob. 19 with wire SH disconnected from point H and connected directly to point R. Use 0.05 ohm as the resistance for wire SR.

Example 4. In Fig. 42, A represents a group of lamps and M represents a motor. The generator supplies 3.7 kw. at 125 volts and the e.m.f. at

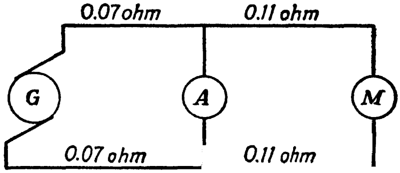


FIG. 42.

the motor is 116 volts. Find the generator current, the voltage at the lamps, the motor current, and the current taken by the lamps.

Solution: 1. Find the generator current.

$$\begin{aligned} W &= EI \\ 3{,}700 &= 125I \\ 1{,}25I &= 3{,}700 \\ I &= 29.6 \text{ amp. generator current} \end{aligned}$$

2. Find the voltage at A.

$$\begin{aligned} &\text{Voltage drop, } G \text{ to } A = 0.14(29.6) = 4.14 \text{ volts} \\ &\text{Voltage at } A = 125 - 4.14 = 120.86 \text{ volts} \end{aligned}$$

3. Find the motor current and the current taken by the lamps.

$$\text{Voltage drop } A \text{ to } M = 120.86 - 116 = 4.86 \text{ volts.}$$

$$\begin{aligned} E &= IR \\ 4.86 &= I(0.22) \\ 0.22I &= 4.86 \\ I &= 22.1 \text{ amp. through } M \end{aligned}$$

The generator current is 29.6 amp.

$$\begin{aligned} \therefore \text{ The current through the lamps } &= 29.6 - 22.1 \\ &= 7.5 \text{ amp.} \end{aligned}$$

22. Draw a diagram like the one in Fig. 42. Mark each line wire between G and A as having a resistance of 0.08 ohm and each wire between A and M as having a resistance of 0.12 ohm. The generator output is 10 kw. at 130 volts and the e.m.f. at the motor is 115 volts. Find the generator current, the voltage across A, the motor current, and the current through A.

23. Repeat Prob. 22 with each line wire between G and A having a resistance of 0.05 ohm and each wire between A and M having a resistance of 0.09 ohm. The generator output is 5 kw. at 220 volts and the e.m.f at the motor is 216 volts.

24. Repeat Prob. 22 with each line wire between G and A having a resistance of 0.07 ohm and each wire between A and M having a resistance of 0.09 ohm. The generator output is 4 kw. at 125 volts and the e.m f. at the motor is 116 volts.

25. Repeat Prob. 22 with each line wire between G and A having a resistance of 0.1 ohm and each wire between A and M having a resistance of 0.15 ohm. The generator delivers 2.2 kw. at 115 volts and the voltage at the motor is 109.

Example 5. A 15-hp. motor whose full-load efficiency is 90 per cent, is receiving power from a generator whose terminal e.m.f. is 225 volts. If the allowable drop in the line is 2.5 per cent of the generator voltage, what is the allowable resistance of the line between motor and generator?

Solution: 1. Allowable voltage drop in the line = 225(0.025) = 5.625 volts.

2. Find the line current.

$$\text{Efficiency} = \frac{\text{output}}{\text{input}}$$

$$0.90 = \frac{15(746)}{x}$$

$$0.90x = 11{,}190$$

$$x = 12{,}433 \text{ watts input to motor}$$

The e.m.f. at the motor is equal to the generator voltage less the line drop.

$\therefore$ The e.m.f. at the motor = 225 − 5 625 = 219.375 volts.

$$W = EI$$

$$12{,}433 = 219.375I$$

$$219.375I = 12{,}433$$

$$I = 56.7 \text{ amp. line current}$$

3. Find the resistance of the line wires.

$$E = IR$$

$$5.625 = 56.7R$$

$$56.7R = 5.625$$

$$R = 0.0992 \text{ ohm } Ans.$$

26. A 10-hp. motor, full-load efficiency 88 per cent, is receiving power from a generator whose terminal e.m.f. is 225 volts. The allowable voltage drop in the line is 2 per cent of the generator voltage. What is the maximum allowable resistance of the line wires between motor and generator?

27. In Prob. 26, if the distance between motor and generator is 440 ft., what size B & S copper line wire should be used?

28. Repeat Prob. 26 for a 7.5-hp. motor, 86 per cent efficient, generator e.m.f. 230, and allowable line drop 2 per cent of generator voltage.

29. In Prob. 28, the distance between motor and generator is 480 ft. What size B & S copper line wire should be used?

30. Repeat Prob. 26 for a 5-hp. motor, efficiency 85 per cent, generator e.m.f 115 volts, and allowable line drop 2.5 per cent of generator voltage.

31. In the preceding problem, what size B & S gage copper line wire should be used if the distance between the machines is 110 ft.?

32. A two-wire copper feeder runs from the busbars of switchboard to a lighting panel on which the load is 40 amp. A smaller feeder connects the first lighting panel to a second panel which carries a load of 35 amp. What size B & S rubber-covered wire should be used between the switchboard and the first panel to satisfy the requirements of the table given in Sec. 51? What size wire should be used between the two panel boxes?

33. In Prob. 32, the distance between the switchboard and the first panel box is 250 ft. The distance to the next panel box is 175 ft. What will be the voltage at each panel if the voltage at the switchboard is 125 volts?

34. In Prob. 11, what size wire is used in the various sections of the line if the distances are as follows: main feeder to A = 120 ft., A to B = 38 ft., main feeder to C = 58 ft., C to D = 48 ft.

35. A generator is supplying 5.6 kw. to a line at the other end of which there is a motor delivering 6.4 hp. Motor efficiency is 88.5 per cent. What size copper wire is used between the two machines? The distance between the machines is 510 ft. and the terminal voltage of the generator is 125 volts.

36. A generator is supplying 3.8 kw. to a line delivering power to a building 1,500 ft. from the generator. The allowable loss is 3 per cent of the power supplied. What size copper wire should be used if the line current is 76 amp.?

37. A motor and a group of lamps are connected in parallel across the end of a power line. The lamps consume 17.6 kw., and the motor is delivering 12 hp. Motor efficiency is 87.5 per cent. Of the total power supplied by the generator at the other end of the line 2.5 per cent is lost in the line wires. What is the current flowing if the resistance of the line is 0.045 ohm?

38. In Prob. 37, what is the generator voltage and what is the voltage at the end of the line?

CHAPTER XVIII

GENERATOR AND MOTOR PROBLEMS

85. E.M.F. of a Generator.—The e.m.f. of a generator is the total voltage developed by the generator armature. The e.m.f. generated may be determined by the formula

$$E = \frac{N\phi S}{10^8}$$

where E is the e.m.f. generated in volts
N is the number of active armature conductors
ϕ is the flux per pole
S is the number of revolutions per second of the armature

This formula is correct when the number of poles is the same as the number of paths through the armature, which is usually the case.

Example 1. What is the e.m.f. developed by a generator having 475 active conductors on its armature, which is rotating at 900 revolutions per minute (r.p.m.)? The flux per pole is 7,200,000 lines.

Solution: $N = 475$, $\phi = 7{,}200{,}000$, $S = {}^{900}/_{60}$

Substituting these values in the formula

$$E = \frac{N\phi S}{10^8}$$

$$E = \frac{475 \times 7{,}200{,}000 \times 900}{100{,}000{,}000 \times 60}$$

$$= \frac{475 \times 72 \times 15}{1{,}000} \text{ by cancellation}$$

$$= \frac{513{,}000}{1{,}000}$$

$$= 513 \text{ volts } \mathit{Ans.}$$

Example 2. Find the number of active conductors on an armature which develops 702 volts, while passing through a flux of 6,750,000 lines per pole, twelve hundred times each minute.

Solution: $E = 702$, $\phi = 6{,}750{,}000$, $S = \dfrac{1{,}200}{60}$.

Substituting these values in the formula gives

$$702 = \frac{N \times 6{,}750{,}000 \times 1{,}200}{100{,}000{,}000 \times 60}$$

$$702 = \frac{N \times 675 \times 2}{1{,}000} \text{ by cancellation}$$

$$2(675)N = 702{,}000 \text{ by cross-multiplying}$$

$$N = \frac{702{,}000}{2 \times 675}$$

$$N = 520 \textit{ Ans.}$$

Problems

1. A generator has 650 active conductors and the flux per pole is 6,500,000 lines. Find the e.m.f. generated if the armature rotates at 1,100 r.p.m. and there are two poles and two brushes.

2. If the speed of the above generator were raised to 1,300 r.p.m., what would the e.m.f. of the generator become?

3. If the flux per pole, in Prob. 1, were increased to 7,000,000 lines, what would be the e.m.f. generated?

4. An armature has a certain number of active conductors. The flux per pole is 6,600,000 lines. The armature rotates at 1,500 r.p.m. and develops an e.m.f. of 770.55 volts. How many active conductors are there on the armature?

5. Find the speed at which an armature is rotating if it develops an e.m.f. of 874 volts, the flux per pole is 5,700,000, and there are 800 active conductors on the armature.

6. A generator develops an e.m.f. of 405 volts with the armature rotating at 900 r.p.m. There are 360 active conductors. What is the flux per pole?

7. An armature develops an e.m.f. of 135 volts. At what speed is it rotating if the flux per pole is 5,000,000 lines and there are 540 active conductors?

8. A generator armature revolving at 160 r.p.m. develops an e.m.f. of 105.6 volts. Find the flux per pole if there are 900 active conductors.

9. Find the e.m f. generated by an armature revolving in a field of 2,000,000 lines per pole. There are 320 active conductors and the speed is 1,200 r.p.m.

10. At what speed is an armature rotating if it generates 592.8 volts? Flux per pole is 5,200,000 and there are 380 active conductors.

11. How many active conductors are there on an armature which rotates at 875 r.p.m. through a field of 2,700,000 lines and develops 191.8 volts.

86. The Series Generator.—A generator whose field winding consists of a relatively few turns of heavy wire connected in series with the armature is called a *series generator.*

In Fig. 43, A represents the armature of a series generator and S the series field. The terminals of the machine are represented by MN.

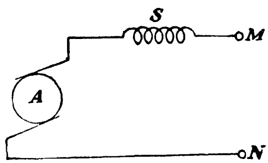


FIG. 43.—Diagram of connections for a series generator.

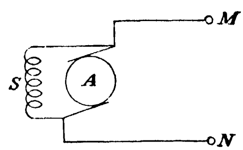
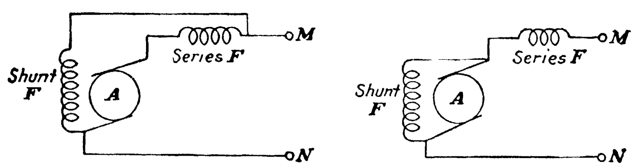


FIG. 44.—Diagram of connections for a shunt generator.

87. The Shunt Generator.—When the field winding of a generator consists of a large number of turns of small-gage wire connected directly across the armature, the machine is known as a *shunt generator.* Figure 44 shows the diagram of connections of a shunt generator.

88. Compound Generators.—A generator which contains both a series and a shunt filed is known as a *compound generator.* There are two ways of connecting the shunt field of a compound generator. These are the *long-shunt* method and the *short-shunt* method. Figure 45 gives an illustration of each.



(a)-Long Shunt Connection (b)-Short Shunt Connection

FIG. 45.—Methods of connecting the shunt field of a compound generator.

Example 3. A compound generator has a shunt field whose resistance is 90 ohms, a series field whose resistance is 0.12 ohm, and an armature of 0.09 ohm resistance. The machine delivers 30 amp. to the line and is connected as a long-shunt generator with its terminal e.m.f. equal to 117 volts. Find the total voltage generated by the armature of this machine.

Figure 46 illustrates the conditions of the problem.

Solution: 1. Find the shunt-field current.

From Fig. 46, it is evident that the shunt field is connected directly across the terminals of the generator, where the e.m.f. is 117 volts.

$$\therefore 117 = 90I$$
$$90I = 117$$
$$I = 1.3 \text{ amp. shunt-field current}$$

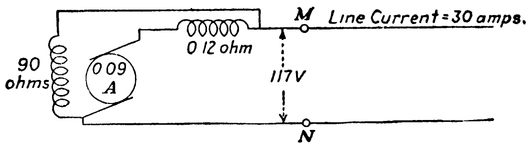


FIG. 46.

2. Find the armature current.

Since the generator supplies all of the current used in this circuit, the shunt-field current, as well as the line current, must pass through the armature of the generator.

$$\therefore \text{The armature current} = 30 + 1.3 = 31.3 \text{ amp.}$$

3. Find the voltage drop in armature and series field.

Since the series field is connected directly in series with the armature, we may add their resistance values.

$$0.09 + 0.12 = 0.21 \text{ ohm}$$

The voltage drop in armature and series field together is 31.3(0.21) = 6.573 volts

4 Find the total e.m.f. generated.

The armature and series field are connected in series with the line. The voltage lost in the armature and the series field is, therefore, in series with the voltage across the line, and the total voltage generated must be the sum of 117 and 6.573.

$$\therefore \text{The total e.m.f. generated} = 117 + 6.573$$
$$= 123.573 \text{ volts}$$

Example 4. A short-shunt compound generator has a series-field resistance of 0.15 ohm, an armature resistance of 0.1 ohm, and a shunt-field resistance of 124 ohms. This generator is delivering 40 amp. to a group of lamps over a line whose resistance is 0.2 ohm. The e.m.f. at the lamps is 110 volts. What is the total e.m.f. developed by this generator?

Solution: 1. Find the voltage at the terminals of the machine (*MN*, Fig. 47).

Line drop, *MN* to lamps = 40(0.2) = 8 volts

Voltage at *MN* = 110 + 8 = 118 volts

2. Find the voltage across the shunt field. The shunt field is connected directly across the armature. If we add to the voltage at MN the voltage drop in the series field, we will obtain the voltage across the armature. This is also the voltage across the shunt field.

$$IR \text{ drop, series field} = 40(0.15) = 6 \text{ volts}$$
$$\text{Voltage across shunt field} = 118 + 6 = 124 \text{ volts}$$

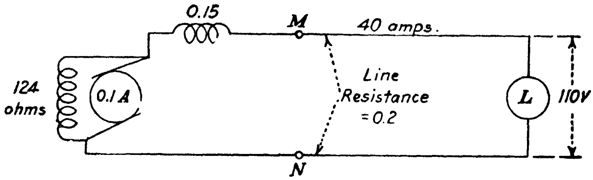


FIG. 47.

3 Find the armature current.

The shunt-field current is found by

$$\begin{aligned} E &= IR \\ 124 &= I(124) \\ 124I &= 124 \\ I &= 1 \text{ amp. shunt-field current} \end{aligned}$$

$\therefore$ The current through the armature is $40 + 1 = 41$ amp.

4. Find the total e.m.f. generated.

The voltage drop in the armature $= 41(0.1) = 4.1$ volts.

The voltage across the armature was found (in step 2) to be 124 volts.

$\therefore$ The total voltage generated $= 124 + 4.1 = 128.1$ volts *Ans.*

Problems

1. The voltage across the terminals of a series generator is 125 volts when the generator is delivering 56 amp. What is the total e.m.f. generated by this machine if the armature resistance is 0.07 and the field resistance 0.05 ohm?

2. A shunt generator is delivering 80 amp. to a line. Terminal voltage of the machine is 120 volts. The shunt field has a resistance of 120 ohms and the armature of 0.075 ohm. What e.m.f. is generated by the armature?

3. A shunt generator whose brush potential is 225 volts is delivering 65 amp. to the line. What is the e.m.f. generated if the armature resistance is 0.05 ohm and the shunt field has a resistance of 150 ohms?

4. A compound generator has a shunt-field resistance of 148 ohms and series-field resistance of 0.05 ohm. The armature resistance is 0.06 ohm. The terminal voltage of the machine is 111 volts and the line current is 42 amp. If the machine is connected as a long-shunt generator. what is the e.m.f. generated?

5. If the short-shunt connections were used in Prob. 4, what e.m.f. would be generated, all other conditions remaining the same?

6. A compound generator has a terminal voltage of 224 volts. It is delivering 37 amp. to the line. The shunt-field resistance is 160 ohms, series-field resistance is 0.1 ohm, and armature resistance is 0.15 ohm If connected short shunt, what is the e.m.f. generated?

7. If the generator in Prob. 6 were connected as a long-shunt machine, all other conditions remaining the same, what would be the total e.m.f. generated?

8. A shunt generator delivers 32 amp. to a load connected by a line whose resistance is 0.45 ohm. The line potential at the load is 114 volts. What is the e.m.f. generated by the armature of the machine, the shunt-field resistance being 125 ohms and the armature resistance 0.15 ohm?

9. A short-shunt compound generator delivers 80 amp. to a group of lamps over a line whose resistance is 0.23 ohm. The e.m.f. at the lamps is 115 volts. The armature resistance is 0.09 ohm, series-field resistance 0.11 ohm, and the shunt-field resistance is 120 ohms. Find the e.m.f. generated.

10. In Prob. 9, suppose the machine to be connected long shunt and all other conditions to remain the same What e.m.f. is generated?

89. Counter E.M.F. of a Motor.—When the armature of a generator revolves through its magnetic field, an e.m.f. is generated. Consequently, when the armature of a motor revolves in its magnetic field, an e.m.f. is also generated. This e.m.f. of the motor armature and the e.m.f. of the line to which the motor is connected are of opposite polarity. The e.m.f. of the motor is, therefore, referred to as the "counter e.m.f." of the motor.

A shunt motor has its armature connected directly across the line, and the total line voltage must be dissipated in the armature. Since the counter e.m.f. of the motor opposes the line e.m.f., the difference of these two voltages is the voltage which is used in forcing the current through the armature. For a shunt motor, therefore, we have the following relation:

$$\text{Counter e.m.f.} = \text{line voltage} - \text{armature } IR \text{ drop}$$

Example 5. A shunt motor has an armature resistance of 0.08 ohm and a field resistance of 92 ohms. The armature current is 30 amp. Find the counter e.m.f. of the motor The field current is 1.25 amp.

Solution: Line voltage $= 1.25(92) = 115$ volts

$$\text{Counter e.m.f.} = 115 - 0.08(30)$$
$$= 115 - 2.4 = 112.6 \text{ volts } \textit{Ans.}$$

Example 6. A shunt motor has a counter e.m.f. of 221 volts. The line current is 23.1 amp. and the shunt-field current 1.6 amp. The field resistance is 140 ohms. Find the armature resistance.

Solution: Since the line wires carry the total current supplied to the motor, the difference between the current in the line and the current in the shunt field is the armature current.

$$\therefore \text{The armature current} = 23.1 - 1.6 = 21.5 \text{ amp.}$$

$$\text{The line voltage} = 1.6(140) = 224 \text{ volts}$$

If

$$R = \text{armature resistance}$$

Then

$$\begin{aligned} 221 &= 224 - 21.5R \\ 21.5R &= 224 - 221 \\ 21.5R &= 3 \\ R &= 0.14 \text{ ohm } \textit{Ans.} \end{aligned}$$

Example 7. A shunt motor uses a total of 2,280 watts, of which 36.3 watts are lost in the armature and 80 watts are lost in the shunt field. whose resistance is 125 ohms. Find the field current, the line current, the armature resistance, and the counter e.m.f. of the motor

Solution: 1. Find the field current.

$$\begin{aligned} W &= I^2R \\ 80 &= I^2(125) \\ 125I^2 &= 80 \\ I^2 &= 0.64 \\ I &= 0.8 \text{ amp. field current} \end{aligned}$$

2. Find the line current.

$$\begin{aligned} \text{Line voltage} &= 0.8(125) = 100 \text{ volts} \\ W &= EI \\ 2{,}280 &= 100I \\ 100I &= 2{,}280 \\ I &\cong 22.8 \text{ amp. line current} \end{aligned}$$

3. Find the armature resistance.

$$\begin{aligned} \text{Armature current} &= 22.8 - 0.8 = 22 \text{ amp.} \\ W &= I^2R \\ 36.3 &= (22)^2R \\ (22)^2R &= 36.3 \\ 484R &= 36.3 \\ R &= 0.075 \text{ ohm armature resistance} \end{aligned}$$

4. Find the counter e.m.f. of the motor.

$$\begin{aligned} \text{Counter e.m.f} &= 100 - 22(0.075) \\ &= 100 - 1.65 = 98.35 \text{ volts} \end{aligned}$$

Problems

1. A shunt motor whose armature resistance is 0.1 ohm is connected to a 110-volt line. The armature current is 10 amp What is the counter e.m.f. of the motor?

2. A shunt motor is drawing 25 amp. from a 115-volt line. The shunt-field resistance is 115 ohms and the armature resistance is 0.15 ohm. What is the counter e.m.f. of the motor?

3. A shunt motor has an armature resistance of 0.12 ohm and a field resistance of 130 ohms. The field current is 1.1 amp. and the armature current 20 amp. What is the counter e.m.f. of the motor?

4. A shunt motor has a counter e.m.f. of 124.5 volts, at a certain speed. The shunt-field current is 1.2 amp. and the line current 18.2 amp. The field resistance is 125 ohms. What is the armature resistance?

5. A shunt motor whose counter e.m.f is 212 volts is drawing 36.5 amp. from the line. The shunt-field current is 1.5 amp. and the field resistance is 144.6 ohms. What is the armature resistance?

6. How much resistance must be connected in series with the armature in Prob. 5 to keep the starting current through the armature down to 50 amp.?

7. How much resistance must be connected in series with a motor armature of 0.25 ohm resistance to keep the starting current from exceeding 35 amp. on a 110-volt circuit?

8. If the motor of Prob. 7 were to be connected to a 220-volt circuit, how much resistance must be placed in series with the armature to keep the starting current the same as before?

9. The counter e.m.f. of a shunt motor is 107. The shunt field has a resistance of 110 ohms and there is 1 amp. flowing through it. The armature resistance is 0.13 ohm. What is the armature current?

10. In Prob. 9, how much power is supplied to the motor? How much of this is lost in the field and how much in the armature?

11. A shunt motor has 2,288 watts supplied to it; 28 watts are lost in the armature and 98 watts are lost in the field. The field resistance is 137.5 ohms. Find the following:

(*a*) The field current
(*b*) The line current
(*c*) The armature resistance
(*d*) The counter e.m.f. of the motor

12. A shunt motor is drawing 7,755 watts from a line. The armature and shunt-field I^2R losses are 115.6 watts and 275 watts, respectively. If the field current is 1.25 amp., find

(*a*) The field resistance
(*b*) The line current
(*c*) The armature resistance
(*d*) The counter e.m.f. of the motor

13. A shunt motor is delivering 11.6 hp., and at this load its efficiency is 85.09 per cent. The I^2R losses in the shunt field and armature are 270 watts and 96.8 watts, respectively. The field resistance is 187.5 ohms. Find

(a) The field current
(b) The line current
(c) The armature resistance
(d) The counter e.m.f. of the motor

14. A shunt motor is drawing 17,380 watts from a line. The armature and shunt-field I^2R losses are 108 watts and 880 watts, respectively. The field current is 1.8 amp. Find

(a) The field resistance
(b) The line current
(c) The armature resistance
(d) The counter e.m f. of the motor

15. A shunt motor is drawing 2,079 watts from a line. The armature and shunt-field I^2R losses are 202 watts and 89.1 watts, respectively. If the field current is 0.9 amp., find

(a) The field resistance
(b) The line current
(c) The armature resistance
(d) The counter e.m.f. of the motor

16. A shunt motor is drawing 6,528 watts from a line. The armature and shunt-field losses are, respectively, 101.4 watts and 288 watts. The field current is 1.2 amp. Answer the same questions as for Prob 15.

17. A shunt motor is drawing 8,085 watts from a line. The armature and shunt-field losses are, respectively, 249.64 watts and 343 watts. The field current is 1.4 amp. Answer the same questions as for Prob. 15.

90. Determination of Efficiency.—We have learned that the efficiency of a machine is the ratio of its output to its input, the ratio usually being expressed in per cent. To determine the actual efficiency of a generator at various loads, we must determine the input and the output of the machine for each load. The output can be easily determined by means of voltmeter and ammeter readings, but the input is usually determined indirectly by adding to the output the various losses in the machine.

The losses in a generator are divided into two classes, copper losses and stray-power losses. The copper losses are the I^2R

losses in the armature and field. The stray-power losses are composed of friction losses and hysteresis and eddy-current losses. The stray-power loss for a machine does not vary to any great extent and may be considered constant at all loads.

One method of determining the efficiency of a shunt generator is the following:

1. Run the machine as a motor without any load and take readings of the input and field current.
2. Measure the resistance of armature and field and calculate the copper loss for each.
3. The difference between the total copper loss and the input to the machine is the stray-power loss.
4. Now run the machine as a generator, apply different loads, and for each load determine the output and the field current.
5. Calculate the efficiency at each load.

Example 8. A shunt generator is operated as a motor without any load. A voltmeter across the terminals of the machine reads 117 volts and an ammeter shows that the total current is 3.2 amp. An ammeter in series with the field reads 1.2 amp. The field and armature resistance are, respectively, 97.5 ohms and 0.15 ohm. Find the total copper loss and the stray-power loss of the machine.

Solution: The armature current = 3.2 − 1.2 = 2 amp.

I^2R loss in shunt field = $(1.2)^2 97.5$ = 140.4 watts

I^2R loss in armature = $(2)^2(0.15)$ = 0.6 watt

Total copper loss = 140.4 + 0.6 = 141 watts

Total power input = 117(3.2) = 374.4 watts

Stray-power loss = 374.4 − 141 = 233.4 watts

Example 9. The machine of Ex. 8 is operated as a generator with a terminal voltage of 117. The field current is kept constant at 1.2 amp. Find the efficiency of the machine at each of the following external load currents: 5, 7, 10, 20, and 25 amp.

Solution: We shall consider the stray-power loss as being 233.4 watts, as determined in Ex. 8.

Since the armature current is the sum of the line current and the field current, the armature currents at the various loads will be 6.2, 8.2, 11.2, 21.2, and 26.2 amp.

Arrange a table for the losses at the different load currents and fill in the values calculated as follows:

Losses							
1	2	3	4	5	6	7	8
I^2R field	I^2R armature	Stray power	Total losses	Line current	Watts output	Total input	Efficiency, per cent
140.4	5.77	233.4	379.6	5	585	964.6	60.6
140.4	10.1	233.4	383.9	7	819	1,202.9	68.1
140.4	18.8	233.4	392.6	10	1,170	1,562.6	74.9
140.4	67.4	233.4	441.2	20	2,340	2,781.2	84.1
140.4	103	233 4	476.8	25	2,925	3,401.8	86.0

The values in column 1 are the shunt-field losses. They are all equal, since the field current has the same value at all loads. It is calculated as in Ex. 8.

The values in column 2 are obtained by taking the square of the armature current at each load and multiplying by the armature resistance. Thus, for a line current of 5 amp., the armature current is 6.2 amp. and

$$\begin{aligned}\text{the armature } I^2R \text{ loss} &= (6.2)^2 0.15\\ &= 38.44(0.15)\\ &= 5.766 \text{ or } 5.77 \text{ watts}\end{aligned}$$

The values in column 4 are obtained by adding the values in the first three columns.

The values in column 6 are obtained by multiplying the line current by the line voltage, which is 117.

The total input (column 7) is the sum of the total losses (column 4) and the output (column 6).

The efficiency is obtained by dividing the output (column 6) by the input (column 7).

Problems

1. A shunt generator is run light as a motor. A voltmeter across the machine reads 110 volts and an ammeter in the line reads 2.5 amp. The field ammeter reads 1.1 amp. The field and armature resistances are 100 ohms and 0.25 ohm, respectively. Find the total copper loss and the stray-power loss of the machine.

2. The machine of Prob. 1 is operated as a generator with its brush potential 110 volts. Assume the field current to be kept constant at 1.1 amp. Find the efficiency of the machine at each of the following line currents: 0, 5, 8, 10, 15, and 20 amp. Arrange your results in the form of a table, as shown in Ex. 9.

3. Repeat Prob. 1, using the following readings: voltmeter 208 volts, line ammeter 2.6 amp., and field ammeter 1.3 amp. Shunt-field resistance is 160 ohms and armature resistance 0.15 ohm.

4. With the machine in Prob. 3 operating as a generator, the following readings are obtained: voltmeter constant at 208 volts, field ammeter constant at 1.3 amp., line ammeter reads 5, 10, 15, 25, 35, and 45 amp. Find the efficiency at each load.

5. The following readings are obtained when operating a shunt generator as in Prob. 1: voltmeter 550 volts, field ammeter 2.5 amp., ammeter in series with armature reads 1.8 amp. The field resistance is 220 ohms and the armature resistance 0.07 ohm. Find the total copper loss and the stray-power loss.

6. The machine in Prob. 5 is operated as a generator. The terminal voltage is kept at 550 and the field current at 2.5 amp. Find the efficiency of the machine at each of the following loads 10, 20, 30 45, 55, and 70 amp.

7. A shunt generator is run light as a motor. A voltmeter across the machine reads 110 volts, and an ammeter in the line reads 2.7 amp. The field ammeter reads 1.2 amp. The armature resistance is 0.35 ohm. Find the total copper loss and stray-power loss of the machine.

8. With the machine operating as a generator, the voltage is kept at 110 volts and the field ammeter at 1.2 amp. The line current is then varied from 0 to 40 amp. Find the efficiency of the machine when its output is 10 amp., 20 amp., 30 amp., 35 amp., and 40 amp.

9. Using the results of Probs. 2, 4, 6, and 8, plot a graph for each, showing how the efficiency of each generator varies with the load.

CHAPTER XIX

BATTERY PROBLEMS

91. The e.m.f. of a cell is its potential on open circuit. A voltmeter connected across the terminals of a cell will not give the same reading when the cell is furnishing current to a circuit as it will when the circuit is opened. Whenever we speak of the e.m.f. of a cell, we mean its voltage on open circuit.

92. The internal resistance of a cell is the resistance of the path from one electrode to the other within the cell. The internal resistance of a cell varies with the type and condition of the cell and must always be considered when calculating the current delivered by a battery. It is in series with the external circuit.

93. Cells in Series and Parallel.—When a group of cells is connected in series, the internal resistance and e.m.f. of the cells are also in series. The total e.m.f. of the group is found by multiplying the e.m.f. of one cell by the number of cells, and the total internal resistance is determined by multiplying the internal resistance of one cell by the number of cells.

When a number of cells are connected in parallel, the e.m.f. of the group will be the same as the e.m.f. of one cell and the internal resistances of the cells must be considered as being connected in parallel. If the cells are all identical, the internal resistance of a group of cells connected in parallel is equal to the resistance of one cell divided by the number of cells in the group.

94. Formula for Battery Problems.—It is quite unnecessary to memorize a different formula for every different arrangement of cells. We need only to know Ohm's law and to keep in mind the facts enumerated in Sec. 91 to 93. A slight modification of Ohm's law is, however, helpful in the solution of problems of this type. The total resistance of the circuit consists of the internal resistance of the battery in series with

the resistance of the external circuit. This is expressed by the formula

$$E = I(r + R)$$

where E is the e.m.f. of the cell or group of cells

I is the current in the circuit

r is the internal resistance of the cell or group of cells

R is the resistance of the circuit outside the battery

Example 1. A cell has an e.m.f. of 1.5 volts and an internal resistance of 0.1 ohm. How much current will it send through a circuit whose resistance is 2.5 ohms?

Solution: $E = 1.5$, $r = 0.1$, and $R = 2.5$

$$\begin{aligned} \therefore 1.5 &= I(0.1 + 2.5) \\ 1.5 &= 2.6I \\ 2.6I &= 1.5 \\ I &= 0.577 \text{ amp. } \textit{Ans.} \end{aligned}$$

Example 2. A battery whose e.m.f. is 5.5 volts delivers 2.5 amp. to a circuit whose resistance is 1.9 ohms. What is the internal resistance of the battery?

Solution: $E = 5.5$, $I = 2.5$, and $R = 1.9$

$$\begin{aligned} \therefore 5.5 &= 2.5(r + 1.9) \\ 5.5 &= 2.5r + 4.75 \\ -2.5r &= 4.75 - 5.5 \\ -2.5r &= -0.75 \\ r &= 0.3 \text{ ohm } \textit{Ans.} \end{aligned}$$

Example 3. Twelve cells, each having an e.m.f. of 1.75 volts and an internal resistance of 0.075 ohm, are connected in series. How much current will this battery send through a resistance of 1.5 ohm?

Solution:

$$\begin{aligned} E &= 1.75(12) = 21 \\ r &= 0.075(12) = 0.9 \\ R &= 1.5 \\ \therefore 21 &= I(0.9 + 1.5) \\ 2.4I &= 21 \\ I &= 8.75 \text{ amp. } \textit{Ans.} \end{aligned}$$

Example 4. If the cells of Ex. 3 were connected in parallel, how much current would they send through a circuit whose resistance is 0.1 ohm?

Solution:

$$\begin{aligned} E &= 1.75 \\ r &= 0.075 \div 12 = 0.00625 \\ R &= 0.1 \\ 1.75 &= I(0.00625 + 0.1) \\ 0.1063I &= 1.75 \\ I &= 16.4 \text{ amp. } \textit{Ans.} \end{aligned}$$

Problems

1. A cell has an e.m.f. of 2.2 volts and an internal resistance of 0.15 ohm. How much current can it send through an external resistance of 5 ohms?

2. How much current would the cell in Prob. 1 send through an external circuit of 0.5 ohm?

3. A cell of 1.5 volts e.m.f. and 0.05 ohm internal resistance is delivering 1.3 amp. What is the resistance of the external circuit?

4. If two of the cells in Prob. 3 were joined in series, how much current would they send through an 8-ohm lamp?

5. What is the internal resistance of a battery which delivers 2 amp. to an external circuit of 2.5 ohms, the battery e.m.f. being 6 volts?

6. A cell having an e.m.f. of 1.5 volts and 0.5 ohm internal resistance, is supplying current to an external circuit of 0 5 ohm resistance. What is the total power used in the circuit? What power is used in the external circuit?

7. The cell of Prob. 6 is connected to a lamp whose resistance is 6 ohms. What is the total power used in the circuit? How much power does the lamp use?

8. The cell of Prob. 6 is connected to an external circuit of 0.1 ohm resistance. What is the total power used in the circuit? What is the total power consumed by the 0.1-ohm resistance?

9. 10 cells each of 1.5 volts and 0.05 ohm internal resistance are connected in series. How much current will they send through a 12-ohm resistance; a 0.5 ohm-resistance?

10. 5 of the cells in Prob. 9, connected in series, send 4.3 amp. through an external resistance. What is the value of the resistance?

11. 6 cells, connected in series, send 4 amp. through a circuit of 1.95 ohms resistance. The internal resistance of each cell is 0.2 ohm. What is the e.m.f. per cell?

12. 6 cells, each having an e.m.f. of 1.75 volts, are connected in series to a group of lamps whose total resistance is 1.25 ohms. What is the internal resistance of each cell, the current in the circuit being 3 amp.?

13. 12 cells, each of 2.2 volts e.m.f. and 0.25 ohm internal resistance, are connected in series. The external circuit consists of two lamps, 36 ohms and 48 ohms resistance, connected in parallel. How much current will each lamp receive?

14. The cells in Prob. 13 are connected to an external circuit consisting of a 50-ohm and a 75-ohm lamp in parallel. How much current will each lamp receive?

15. 5 cells, each having an e.m.f. of 1.8 volts and an internal resistance of 0.06 ohm, are connected in parallel. How much current will flow through an external circuit of 1 ohm?

16. If 8 cells like those in Prob. 15 were connected in parallel, how much current would they send through an external circuit of 6 ohms?

17. If the cells of Prob. 9 were connected in parallel, how much current would they send through a 12-ohm resistance?

18. A cell has an e.m.f. of 1.5 volts. When delivering 5 amp. to an external circuit, a voltmeter placed across the cell reads 1.15 volts. What is the internal resistance of the cell and what is the resistance of the external circuit?

19. A cell has an e.m.f. of 2.1 volts on open circuit. When delivering 0.6 amp. to a group of lamps, the voltage across the cell drops to 1.92 volts. Find the internal resistance of the cell and the resistance of the lamps.

20. A radio tube draws 0.25 amp. when it has 5 volts across its terminals. What size variable rheostat should be placed in series with the tube when operating on a 6-volt battery to keep the current at 0.25 amp.?

21. A radio tube which is to operate at 0.06 amp. is connected across a battery of 4.5 volts. When this tube has 3 volts across its terminals, the current is 0.06 amp. What size variable rheostat should be used with the tube?

22. What size variable rheostat should be used with a tube which is rated 1 volt, 0.25 amp., if it is to be used with a cell whose e.m.f. is 1.5 volts?

Example 5. A cell has an e.m.f. of 6.5 volts and an internal resistance of 0.12 ohm. A certain number of these cells, when connected in series, force a current of 15 amp. through a circuit of 10.34 ohms resistance. How many cells are there in the group?

Solution: Let x = the number of cells

$$\begin{aligned} \text{Then} \qquad E &= 6.5x \\ \text{And} \qquad r &= 0.12x \\ \therefore 6.5x &= 15(0.12x + 10.34) \\ 6.5x &= 1.8x + 155.1 \\ 4.7x &= 155.1 \\ x &= 33 \text{ cells } \textit{Ans.} \end{aligned}$$

Example 6. A cell has an e.m.f. of 6.5 volts and an internal resistance of 0.2 ohm. How many of these cells must be connected in parallel to send a current of 15 amp. through a circuit whose resistance is 0.4 ohm?

Solution: Let x = the number of cells

$$\text{Then} \qquad E = 6.5$$

$$\text{And} \qquad r = \frac{0.2}{x}$$

$$\therefore 6.5 = 15\left(\frac{0.2}{x} + 0.4\right)$$
$$6.5 = \frac{3}{x} + 6$$
$$6.5x = 3 + 6x$$
$$0.5x = 3$$
$$x = 6 \text{ cells } \textit{Ans.}$$

Example 7. A cell has an e.m.f. of 1.6 volts and an internal resistance of 0.08 ohm. Of these cells, 51 are arranged in three parallel groups, each group consisting of 17 cells connected in series. How much current will this battery send through a circuit whose resistance is 3.15 ohms?

Solution: The e.m.f. of 17 cells in series $= 1.6(17) = 27.2$ volts

The internal resistance of 17 cells in series $= 0.08(17) = 1.36$ ohm

Therefore, the e.m.f. of each of the three parallel groups is 27.2 volts and the internal resistance of each group is 1.36 ohm. For the entire battery of 51 cells we have:

$$E = 27.2$$
$$r = \frac{1.36}{3} = 0.453$$
$$\therefore 27.2 = I(0.453 + 3.15)$$
$$3.603I = 27.2$$
$$I = 7.55 \text{ amp. } \textit{Ans.}$$

23. A certain number of cells, each having an e.m.f. of 1.5 volts and an internal resistance of 0.05 ohm, are connected in series and force 5 amp. through a 2.25-ohm lamp. How many cells are there in the group?

24. A group of cells, each having an e.m.f. of 2.2 volts and 0.4-ohm internal resistance, when connected in series, force 1.1 amp. through a 6.4-ohm resistance. How many cells are there in the group?

25. A certain number of the cells described in Prob. 23, when connected in parallel, force 2.5 amp. through an external circuit of 0.59 ohm resistance. How many cells are there in the group?

26. A certain number of the cells described in Prob. 24, when connected in parallel, send a current of 5 amp. through a resistance of 0.415 ohm. How many cells are there in the group?

27. A group of cells, each having an e.m.f. of 2.1 volts and an internal resistance of 0.3 ohm, when connected in series, send a current of 0.015 amp. through an external circuit of 8,382 ohms resistance. How many cells are there in the group?

28. A group of cells like those described in the preceding problem are connected in parallel. How many cells are in the group, if the current through the 0.0225-ohm external circuit is 35 amp.?

29. A number of cells, connected in series, force a current of 6 amp. through a circuit whose resistance is 4.95 ohms. Find the number of

cells, if each of them has an e.m.f. of 1.4 volts and an internal resistance of 0.05 ohm.

30. A group of cells connected in parallel sends 40 amp. through a circuit whose resistance is 0.033 ohm. How many cells are in the group if each has an e.m.f. of 1.4 volts and an internal resistance of 0.05 ohm?

31. A cell has an e.m.f. of 2.11 volts and an internal resistance of 0.02 ohm. How many of these cells, connected in parallel, will be needed to send 14 amp. through a circuit whose resistance is 0.15 ohm?

32. A certain number of cells, each having an e.m.f. of 2.24 volts and an internal resistance of 0.08 ohm, are connected in series. The current through an external circuit of 4 ohms is 14 amp. How many cells are there?

33. A certain number of cells in parallel are connected to a circuit whose resistance is 0.104 ohm. The e.m.f. per cell is 2.2 volts and the internal resistance per cell is 0.08 ohm. Find the number of cells. The current in the circuit is 21 amp.

34. 20 cells, each having an e.m.f. of 1.5 volts and an internal resistance of 0.05 ohm, are arranged in 4 groups, each group consisting of 5 cells in series. How much current would this battery send through a circuit of 2.65 ohms?

35. How much current would flow in the external circuit of Prob. 34, if the cells were arranged in 5 groups of 4 cells each?

36. How much current would flow if the cells in Prob. 34 were arranged in 2 groups of 10 cells each?

37. 32 cells, each having an e.m.f. of 2.2 volts and an internal resistance of 0.15 ohm, are arranged in 4 groups of 8 cells each. How much current will they send through a group of lamps whose resistance is 38.5 ohms?

38. How much current would the cells of Prob. 37, arranged in 8 groups of 4 cells each, send through an external resistance of 0.125 ohm?

39. How much current would the cells of Prob. 37, arranged in 2 groups of 16 cells each, send through an external resistance of 7.6 ohms?

40. 48 cells, each having an e.m.f. of 1.3 volts and an internal resistance of 0.4 ohm, are arranged in 4 groups of 12 cells each. How much current will these cells send through an external circuit of 4.56 ohms resistance?

41. How much current would the cells of Prob. 40, arranged in 6 groups of 8 cells each, send through a circuit whose resistance is 0.75?

42. How much current will the cells of Prob. 40, arranged in 2 groups of 24 cells each, send through a circuit whose resistance is 95.2 ohms?

43. How much current would the cells of Prob. 40, arranged in 8 groups of 6 cells each, send through a circuit whose resistance is 1.2 ohms?

44. A 3-cell storage battery is to be charged from 110-volt direct-current mains. Each cell has an e.m.f. of 1.85 volts and an internal resistance of 0.03 ohm. How much resistance must be placed in series with the battery to keep the initial current at 8 amp.?

45. In Prob. 44, how much power is used in charging the battery and how much is lost in the resistance?

46. What must be the voltage of a generator which is to charge 100 lead cells connected in series. The maximum e.m.f. of each cell is 2.2 volts and the average internal resistance is 0.02 ohm per cell. The charging rate is 15 amp.

47. A group of 60 cells is to be charged from a 110-volt line. Each cell has an e.m.f. of 1.05 volts and an internal resistance of 0.025 ohm. How much resistance must be placed in series with the cells if they are connected in series and the charging current should not exceed 12 amp.?

48. 360 cells of the type in the preceding problem are to be charged from a 110-volt line. If the cells are arranged in 5 parallel groups, how much resistance must be connected in series with them if the charging current is not to exceed 14 amp. per cell?

49. A battery of 120 storage cells, each having an e.m.f. of 1.85 volts and an average internal resistance of 0.015 ohm, is to be charged from a 115-volt line. The cells are arranged in three parallel groups. How much resistance must be connected in series with the batteries the maximum charging rate being 15 amp. per cell?

CHAPTER XX

SOLUTION OF LITERAL EQUATIONS

95. Literal Equations.—Equations in which some or all of the known quantities are represented by letters are known as *literal equations.* Formulas such as are used in mathematics and other sciences are literal equations. They may be expressed in more than one form. When we have stated an equation in such a way that the value of one of the quantities in the equation, such as x, is given in terms of all the other quantities, we say that the equation has been solved for x.

Example 1. In the formula $\frac{1}{R} = \frac{1}{a} + \frac{1}{b}$ solve for R.

Solution: Multiply the equation by abR, which is the L.C.D. This gives

$$\frac{1(abR)}{R} = \frac{1(abR)}{a} + \frac{1(abR)}{b}$$

Cancelling gives

$$ab = bR + aR$$

Next, bring all terms containing R, the quantity for which we are to solve, to the left-hand member of the equation. This gives

$$-aR - bR = -ab$$

For convenience, change all signs and then factor the left-hand member. We then have

$$R(a + b) = ab$$

Divide both sides of the equation by $a + b$. This gives

$$\frac{R(a + b)}{a + b} = \frac{ab}{a + b}$$

Now cancel the two $(a + b)$ expressions, and we then have

$$R = \frac{ab}{a + b} \; Ans.$$

Example 2. In the formula $E = N\phi S/10^8$, solve for S.

Solution: Multiply both members of the equation by 10^8, which is the L.C.D. This gives

$$E(10^8) = \frac{N\phi S(10^8)}{10^8}$$

$$10^8E = N\phi S \text{ by cancellation.}$$

Now bring the term containing S to the left, and change signs

$$N\phi S = 10^8E$$

Divide both sides by $N\phi$, the coefficient of S

$$\frac{N\phi S}{N\phi} = \frac{10^8E}{N\phi}$$

Then, by cancellation

$$S = \frac{10^8E}{N\phi} \; Ans.$$

Problems

In each of the following formulas, solve for the quantity or quantities indicated.

1. $W = EI$. Solve for E.

2. $f = \frac{PS}{120}\cdot$ Solve for S.

3. $X = 2\pi fL$. Solve for L.

4. $X = \frac{1}{2\pi fC}\cdot$ Solve for C and f.

5. $Z^2 = R^2 + X^2$. Solve for R.

6. $W = EI \cos \theta$. Solve for E and I.

7. $W = I^2R$. Solve for I and R.

8. $\frac{f_1}{f_2} = \frac{X_1}{X_2}\cdot$ Solve for X_1.

9. $g = \frac{R}{R^2 + X^2}\cdot$ Solve for X.

10. $b = \frac{X}{R^2 + X^2}\cdot$ Solve for R.

11. $\cos A = \frac{b}{c}\cdot$ Solve for c. ($\cos A$ is one quantity.)

12. $\sin A = \frac{a}{c}\cdot$ Solve for a. ($\sin A$ is one quantity.)

13. $\frac{1}{c} = \frac{1}{c_1} + \frac{1}{c_2}\cdot$ Solve for c.

14. Using the formula for Prob. 13, solve for c_1.

15. Using the formula for Prob. 13, solve for c_2.

16. $\frac{1}{R} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\cdot$ Solve for R.

17. Using the formula for Prob. 16, solve for b.

18. $C = \frac{5}{9}(F - 32)$. Solve for F.

19. $I = \dfrac{en}{R + nr}$. Solve for R and n.

20. $R = \dfrac{kl}{d^2}$. Solve for d.

21. Using the formula for Prob. 20, solve for l.

22. $A = \dfrac{h}{2}(a + b)$. Solve for a.

23. $\dfrac{1}{C} = \dfrac{1}{c_1} + \dfrac{1}{c_2} + \dfrac{1}{c_3}$. Solve for C.

24. Using the formula for Prob. 23, solve for c_3.

25. $\dfrac{R_1A_1}{L_1} = \dfrac{R_2A_2}{L_2}$. Solve for A_1.

26. Using the formula for Prob. 25, solve for R_2.

27. $\dfrac{R_1d_1^2}{L_1} = \dfrac{R_2d_2^2}{L_2}$. Solve for R_1.

28. Using the formula for Prob. 27, solve for d_2.

29. $E_C = E_L - I_AR_A$. Solve for R_A.

The following problems should not be attempted until the work on simultaneous equations, Chap. XXI, has been completed.

30. Using both equations solve for W in terms of I and R.

$$E = IR$$
$$W = EI$$

31. Using the formulas given in Prob. 30, solve for E in terms of W and R.

32. Using both equations solve for I in terms of E, k, l, and d^2.

$$E = IR$$
$$R = \frac{kl}{d^2}$$

33. Using both equations solve for E in terms of I, R, and $\cos\theta$.

$$E = IZ$$
$$R = Z\cos\theta$$

34. Using both equations solve for R in terms of X, $\sin\theta$, and $\cos\theta$.

$$X = Z\sin\theta$$
$$R = Z\cos\theta$$

35. Using the equations given in Prob. 34, solve for $\sin\theta$ in terms of X, R, and $\cos\theta$.

36. Using both equations solve for R in terms of W, E, and Z.

$$W = I^2R$$
$$E = IZ$$

37. Using the equations given in Prob. 36, solve for E in terms of W, R, and Z.

CHAPTER XXI

SOLUTION OF SIMULTANEOUS EQUATIONS

96. Linear Equations.—An equation whose graph is a straight line is called a *linear equation.* This kind of equation contains two unknown quantities, as x and y, only the first power of each unknown being present in the equation. Such an equation is also called an *equation of the first degree.*

97. A system of equations is a group of equations in which each letter has the same value in every equation in the group. The equations comprising such a group are also known as *simultaneous equations.*

98. Solution of Simultaneous Equations.—There are several methods of solving simultaneous equations. The two most commonly used are the method of addition or subtraction and that of substitution.

99. Method of Addition or Subtraction.—In all methods of solving equations containing two unknowns, the object is to eliminate one of the unknowns, thus obtaining an equation which contains only one unknown quantity. The problems in this section are to be solved by the method of addition or subtraction; which is explained in the two examples which follow.

Example 1. Solve the equations

$$2x + 3y = 12 \qquad (1)$$
$$3x - y = 7 \qquad (2)$$

Solution: The first step is to multiply one or both of the equations by numbers which make the coefficients of x or y equal in both equations.

$$2x + 3y = 12 \qquad (1)$$
$$\underline{9x - 3y = 21} \text{ by multiplying Eq. (2) by 3} \qquad (3)$$
$$11x = 33 \text{ adding Eqs. (1) and (3)}$$
$$x = 3$$

Substitute $x = 3$ in either of the two original equations.

$$\begin{aligned} 2(3) + 3y &= 12 && (1) \\ 6 + 3y &= 12 \\ 3y &= 12 - 6 \\ 3y &= 6 \\ y &= 2 \end{aligned}$$

$\therefore x = 3,\ y = 2$ are the required roots.

Check: Substitute these values in Eqs. (1) and (2)

$$\begin{aligned} 6 + 6 &= 12 && (1) \\ 9 - 2 &= 7 && (2) \end{aligned}$$

Example 2. Solve for x and y

$$\begin{aligned} 3x + 5y &= -7 && (1) \\ 5x + 2y &= 1 && (2) \end{aligned}$$

Solution:

$$\begin{aligned} 15x + 25y &= -35 && \text{multiplying Eq. (1) by 5} && (3) \\ 15x + 6y &= 3 && \text{multiplying Eq. (2) by 3} && (4) \\ \hline 19y &= -38 && \text{subtracting Eq. (4) from Eq. (3)} \\ y &= -2 \end{aligned}$$

Substituting $y = -2$ in Eq. (1), we have

$$\begin{aligned} 3x + 5(-2) &= -7 \\ 3x - 10 &= -7 \\ 3x &= 3 \\ x &= 1 \end{aligned}$$

$\therefore x = 1,\ y = -2$ are the required roots.

Check:

$$\begin{aligned} 3 - 10 &= -7 && (1) \\ 5 - 4 &= 1 && (2) \end{aligned}$$

Problems

Solve each of the following systems of equations and check the results:

1. $x + 2y = 5$
   $5x - 2y = 1$
2. $4x + 3y = 24$
   $5x + y = 19$
3. $6x - 2y = 34$
   $5x + y = 39$
4. $4x - 3y = -6$
   $x - 2y = -14$
5. $7x - 3y = 35$
   $9x - 2y = 58$
6. $2x + 5y = 69$
   $3x - 2y = -20$
7. $6x + 3y = 6$
   $11x + 5y = 9$
8. $5h - 3k = 34$
   $15h + 12k = 39$
9. $27r - 32s = 42$
   $16r - 9s = -5$
10. $\frac{x}{2} + \frac{y}{4} = 0$
    $\frac{x}{4} - \frac{y}{2} = -5$

11. $\frac{4x}{5} - \frac{3y}{4} = \frac{3}{20}$

$\frac{5x}{4} + \frac{9y}{8} = 1$

12. $16s - 6t = 1$
$12s + 8t = 7$

13. $\frac{11x}{3} + \frac{5y}{6} = -1$

$\frac{13x}{6} - \frac{3y}{8} = -11$

14. $\frac{5x}{6} - \frac{3y}{7} = \frac{211}{42}$

$\frac{3x}{4} - \frac{2y}{5} = \frac{91}{20}$

15. $9n + 8r = 12$
$12n + 20r = 23$

16. $8m - 6t = -2$
$12m + 11t = 9$

17. $\frac{11x}{6} + \frac{4y}{5} = 17$

$\frac{8x}{3} - \frac{3y}{10} = 54$

18. $15x + 5y = 8$
$4x + 14y = 11$

19. $70r + 9t = 14$
$50r + 5t = 9$

20. $\frac{10x}{3} - \frac{8y}{9} = \frac{2}{10}$

$\frac{13x}{9} - \frac{5y}{3} = \frac{-16}{15}$

21. $0.3r + 0.5N = 3.0$
$0.7r + 0.8N = 5.9$

22. John buys six electric bells and ten 60-watt lamps for \$6.00. If he had bought 5 bells and 4 lamps, he would have paid \$3.70. What was the cost of each article purchased?

23. Five electric irons and 3 heaters together cost \$46.00. Eight irons and 5 heaters cost \$75.00. Find the cost of each.

Example 3. Eight cells connected in series are supplying 6 amp. to a circuit whose resistance is 2.32 ohms. Four of the same cells, connected in parallel, will force 5 amp. through a 0.405-ohm resistance. Find the e.m.f. and internal resistance per cell.

Solution: Let e = the e.m.f. per cell
And r = the internal resistance per cell

Then the e.m.f. of 8 cells in series $= 8e$
And the internal resistance of 8 cells in series $= 8r$
The e.m.f. of 4 cells in parallel $= e$
The internal resistance of this group $= \frac{r}{4}$
Substitute these values in the formula $E = I(r + R)$.

$$8e = 6(8r + 2.32) \text{ series connection} \qquad (1)$$

$$e = 5\left(\frac{r}{4} + 0.405\right) \text{ parallel connection} \qquad (2)$$

$$e = \frac{5r}{4} + 2.025$$

$$4e = 5r + 8.1 \text{ by clearing of fractions} \qquad (3)$$

$$8e = 48r + 13.92 \text{ multiplying in Eq. (1)} \qquad (4)$$

$$\underline{8e = 10r + 16.2 \text{ multiplying Eq. (3) by 2}} \qquad (5)$$

$$0 = 38r - 2.28 \text{ subtracting Eq. (5) from Eq. (4)}$$

$$-38r = -2.28$$

$$r = 0.06 \text{ ohm}$$

Substitute this value of r in Eq. (1), and we have

$$8e = 6(0.48 + 2.32)$$

$$8e = 6(2.8)$$

$$8e = 16.8$$

$$e = 2.1 \text{ volts}$$

$$\left.\begin{array}{ll}\text{The internal resistance per cell} & = 0.06 \text{ ohm} \\ \text{And the e.m.f. per cell} & = 2.1 \text{ volts}\end{array}\right\} \textit{Ans.}$$

24. Ten cells when joined in series send a current of 2 amp. through a resistance of 6.5 ohms. The same cells, connected in parallel, send a current of 3 amp. through a resistance of 0.49 ohm. Find the e.m.f. and the internal resistance per cell.

25. Five cells, when connected in series, send a current of 3 amp. through a resistance of 6 ohms. The same cells, when connected in parallel, send a current of 7 amp. through a resistance of 0.8 ohm. Find the e.m.f. and the internal resistance per cell.

26. Eight cells, when connected in series, send a current of 1.6 amp. through an external resistance of 2.8 ohms. When connected in parallel the same cells send a current of 6 amp. through a circuit whose resistance is 0.15 ohm. Find the e.m.f. and internal resistance per cell.

27. Five cells, joined in parallel, force a current of 7.5 amp. through an external resistance of 0.19 ohm. The same cells, when connected in series, force a current of 10 amp. through a resistance of 0.50 ohm. Find the e.m.f. and internal resistance per cell.

28. Eight cells in series send 4.3 amp. through a circuit whose resistance is 2.72 ohms. Five of the same cells, when connected in parallel, send 2.5 amp. through a circuit of 0.828 ohm resistance. Find the e.m.f. and internal resistance per cell.

29. Twenty cells in series send 0.12 amp. through a circuit whose resistance is 295 ohms. Five of the same cells, when connected in parallel, send 1.8 amp. through 0.95 ohm. Find the e.m.f. and internal resistance per cell.

30. Twelve cells in series send 0.7 amp. through a circuit of 29.88 ohms. Six of the same cells, connected in parallel, send 3.5 amp. through a circuit of 0.5 ohm. Find the e.m.f. and internal resistance per cell.

31. Six cells, connected in series, force a current of 3 amp. through a circuit of 1.8 ohms resistance. Five of these cells, connected in parallel to a circuit of 0.24 ohm resistance, cause a current of 5 amp. to flow. Find the e.m.f. and internal resistance per cell.

32. Five cells, connected in parallel to a circuit whose resistance is 1.04 ohm, cause 2 amp. to flow. Ten of the same kind of cells, connected in series, send 6 amp. through a 3-ohm circuit. Find the e.m.f. and internal resistance per cell.

100. Method of Substitution.—When the value of one of the unknowns in a system of equations can be readily expressed in terms of the other, the method of substitution can be conveniently used

Example 4 Solve by the method of substitution

$$x + 5y = 22 \qquad (1)$$
$$3x - 2y = -2 \qquad (2)$$

Solution

$$x + 5y = 22 \qquad (1)$$
$$x = 22 - 5y$$

Substituting this value of x in Eq (2) gives

$$3(22 - 5y) - 2y = -2$$
$$66 - 15y - 2y = -2$$
$$-15y - 2y = -2 - 66$$
$$-17y = -68$$
$$y = 4$$

Substitute $y = 4$ in Eq (1), and we have

$$x + 5(4) = 22$$
$$x + 20 = 22$$
$$x = 2$$

$x = 2$, $y = 4$ are the required roots

Check

$$2 + 20 = 22 \qquad (1)$$
$$6 - 8 = -2 \qquad (2)$$

Example 5 Solve by the method of substitution

$$5x + 2y = 5 \qquad (1)$$
$$2x - 3y = 21 \qquad (2)$$

Solution

$$5x + 2y = 5 \qquad (1)$$
$$5x = 5 - 2y$$
$$x = \frac{5 - 2y}{5}$$

Substituting this value of x in Eq (2) gives

$$2\left(\frac{5 - 2y}{5}\right) - 3y = 21$$
$$\frac{10 - 4y}{5} - 3y = 21$$
$$10 - 4y - 15y = 105$$
$$-4y - 15y = 105 - 10$$
$$-19y = 95$$
$$y = -5$$

Substitute $y = -5$ in Eq. (2) and we have

$$2x - 3(-5) = 21$$
$$2x + 15 = 21$$
$$2x = 6$$
$$x = 3$$

$\therefore x = 3,\ y = -5$ are the required roots.

Check:
$$15 - 10 = 5 \qquad (1)$$
$$6 + 15 = 21 \qquad (2)$$

Problems

Solve each of the following systems of equations by the method of substitution, and check your results:

1. $x + y = 9$
   $5x + y = 17$
2. $4x + y = 17$
   $7x - 2y = 11$
3. $x + 8y = 4$
   $3x - 5y = -17$
4. $3x - 2y = 0$
   $x - 5y = 13$
5. $4x + 5y = 32$
   $3x - y = -14$
6. $13x - 8y = 10$
   $x + 2y = 40$
7. $4x + 7y = 23$
   $9x + y = 96$
8. $7x - 9y = -13$
   $x - 4y = 9$
9. $3x + 4y = 26$
   $2x + 3y = 19$
10. $5x - 2y = 4$
    $7x - 3y = 4$
11. $5x + 6y = 63$
    $2x - 5y = 3$
12. $3x - 8y = 4$
    $4x + 3y = -22$

101. Simultaneous Equations Containing Three or More Unknown Quantities.—In equations of this type, when each of the unknown quantities appears in every equation, the solution is most easily obtained by the method of addition or subtraction; when this is not the case, the method of substitution can frequently be used to advantage.

Example 6. Solve

$$2x + 3y - 4z = -4 \qquad (1)$$
$$3x - 4y + 3z = 5 \qquad (2)$$
$$4x + y - z = 7 \qquad (3)$$

Solution:
$$6x + 9y - 12z = -12 \text{ multiplying Eq. (1) by 3} \qquad (4)$$
$$12x - 16y + 12z = 20 \text{ multiplying Eq. (2) by 4} \qquad (5)$$
$$18x - 7y = 8 \text{ by adding Eqs. (4) and (5)} \qquad (6)$$

We now have one equation which does not contain z. Next, we take Eq. (3), which has not yet been used, with either Eq. (1) or Eq. (2), and

eliminate z in order to obtain a second equation which contains only x and y.

$$\begin{array}{rlr} 3x - 4y + 3z = 5 & & (2) \\ \underline{12x + 3y - 3z = 21} & \text{multiplying Eq. (3) by 3} & (7) \\ 15x - y \quad = 26 & \text{by adding Eqs. (2) and (7)} & (8) \end{array}$$

Next, take Eqs. (6) and (8) and solve for x and y.

$$\begin{array}{rlr} 18x - 7y = 8 & & (6) \\ \underline{105x - 7y = 182} & \text{multiplying Eq. (8) by 7} & (9) \\ -87x = -174 & \text{by subtracting Eq. (9) from Eq. (6)} & \\ x = 2 & & \\ 30 - y = 26 & \text{by substituting } x = 2 \text{ in Eq. (8)} & \\ -y = 26 - 30 & & \\ y = 4 & & \end{array}$$

Substitute $x = 2$ and $y = 4$ in Eq. (2), and we have

$$6 - 16 + 3z = 5$$
$$3z = 5 - 6 + 16$$
$$3z = 15$$
$$z = 5$$
$$\therefore x = 2,\ y = 4,\ z = 5 \text{ are the required roots}$$

Check:

$$4 + 12 - 20 = -4 \qquad (1)$$
$$6 - 16 + 15 = 5 \qquad (2)$$
$$8 + 4 - 5 = 7 \qquad (3)$$

Example 7. Solve

$$5a - 2b = 29 \qquad (1)$$
$$3a + 2c = 39 \qquad (2)$$
$$4b + 7d = 13 \qquad (3)$$
$$2a + 3b + c = 16 \qquad (4)$$

Solution:

$$5a - 2b = 29 \qquad (1)$$
$$-2b = 29 - 5a$$
$$b = \frac{29 - 5a}{-2}$$
$$3a + 2c = 39 \qquad (2)$$
$$2c = 39 - 3a$$
$$c = \frac{39 - 3a}{2}$$

Substitute these values of b and c in Eq. (4), and we have

$$2a + 3\left(\frac{29 - 5a}{-2}\right) + \frac{39 - 3a}{2} = 16$$
$$2a + \frac{87 - 15a}{-2} + \frac{39 - 3a}{2} = 16$$
$$4a - 87 + 15a + 39 - 3a = 32 \text{ by clearing of fractions}$$
$$4a + 15a - 3a = 32 + 87 - 39$$
$$16a = 80$$
$$a = 5$$

$$b = \frac{29 - 5a}{-2}$$

$$\therefore b = \frac{29 - 25}{-2} \text{ by substituting } a = 5$$

$$b = \frac{4}{-2}$$

$$b = -2$$

$$c = \frac{39 - 3a}{2}$$

$$\therefore c = \frac{39 - 15}{2}$$

$$c = {}^{24}\!/_{2}$$

$$c = 12$$

$$-8 + 7d = 13 \text{ by substituting } b = -2 \text{ in Eq. (3)} \qquad (3)$$

$$7d = 21$$

$$d = 3$$

$\therefore a = 5,\ b = -2,\ c = 12,\ d = 3$ are the required roots.

Problems

Solve and check each of the following systems of equations:

1.
$$\begin{aligned} x + y + z &= 6 \\ 2x + y + 2z &= 10 \\ 2x + y - 2z &= 6 \end{aligned}$$

2.
$$\begin{aligned} 2x + y - w &= 2 \\ 3x - y + w &= 8 \\ x + 2y + 3w &= 28 \end{aligned}$$

3.
$$\begin{aligned} 4a + b + c &= 4 \\ 5a + 2b + c &= 6 \\ 3a - 2b + 4c &= 11 \end{aligned}$$

4.
$$\begin{aligned} 4a + 5b &= 71 \\ 3b - 2c &= 11 \\ 2a + 3c &= 33 \end{aligned}$$

5.
$$\begin{aligned} 5a + 2b - c &= 3 \\ 3a + 3b + 2c &= 11 \\ 7a + 3c &= 49 \end{aligned}$$

6.
$$\begin{aligned} 6E - 4R + 5I &= 10 \\ 3E + 2R &= 60 \\ 5E + 4I &= 58 \end{aligned}$$

7.
$$\begin{aligned} 3x + y &= 10 \\ 2x + z &= 9 \\ 5x + w &= 11 \\ 2y + 3z + 4w &= 27 \end{aligned}$$

8.
$$\begin{aligned} 3a + 4b &= 34 \\ 2b + 5c &= 59 \\ 5c + 2d &= 55 \\ 4a + 7d &= 78 \end{aligned}$$

9.
$$\begin{aligned} 6a + b &= 5 \\ 5a + c &= -5 \\ 3a + 4b - 3c &= 5 \\ 8a - 2c - 3d &= -5 \end{aligned}$$

10.
$$\begin{aligned} 5r + s &= 7 \\ 4r + t &= 12 \\ 2r + 5t - 3w &= 6 \\ 5r + 6s + 4w &= 16 \end{aligned}$$

CHAPTER XXII

KIRCHHOFF'S LAWS

102. Kirchhoff's Laws.—Ohm's law may be stated in several different ways, and Kirchhoff's laws are extensions of Ohm's law or different ways of expressing facts which become evident from Ohm's law. These facts may be stated as follows:

1. At any branch point in a circuit, there is as much current flowing away from the point as there is flowing toward it.
2. In any closed circuit, or in any closed portion of a circuit, the e.m.f. applied to the circuit is equal to the algebraic sum of the IR drops in the circuit.

The above statements enable us to write a group of equations for any circuit and to solve the equations for the quantities which are unknown. This simplifies the solution of complicated networks.

103. Conventions Used in Applying Kirchhoff's Laws.—Strict adherence to the following conventions will assist the student greatly in arriving at a solution for the problems in this chapter:

1. Indicate by plus and minus signs the polarity of the given e.m.f. and show by means of an arrow the direction of this e.m.f. The arrow should point from minus to plus. In this manner, mark each of the given e.m.fs.
2. Indicate by means of arrows the direction of current flow in the rest of the circuit. If you are in doubt about the direction of current flow in any branch, place the arrow as you think best. If, in solving a problem, a negative value is obtained for any current, its direction is opposite to that indicated by the arrow, but its numerical value is the same as calculated.
3. When writing the equation for a closed circuit, begin with the e.m.f. in this circuit and pass around the circuit in

the direction in which the e.m.f. is assumed to act. The e.m.f. is then placed equal to the algebraic sum of the IR drops in the circuit. When passing through a resistance in the direction of the current, mark its IR drop positive; and when passing through a resistance in the direction opposed to the direction of current flow, mark its IR drop negative.

4. When a closed portion of the circuit under consideration contains no source of e.m.f., begin at any point and pass around the circuit in any direction. The sum of all the IR drops obtained when passing through resistances in the same direction with the arrows is equal to the sum of all the IR drops obtained when passing through resistances in the direction opposed to that indicated by the arrows.

5. When a loop (closed circuit) contains more than one e.m.f., the algebraic sum of the e.m.fs. must be placed equal to the algebraic sum of the IR drops in the loop.

In the following examples, Kirchhoff's laws are used to solve some simple problems, such as those in Chaps. V and VI. The purpose of this is to teach the use of these laws in solving problems with which the student is already familiar, so that their application in the more difficult problems will be made easier.

Example 1. Three resistances of 7 ohms, 21 ohms, and 30 ohms are connected in series across a 115-volt circuit. Find the current in the circuit.

Solution: In this problem, there is only one closed circuit. Let I represent the current, and use Kirchhoff's second law to write the equation

$$\begin{aligned} 115 &= I(7) + I(21) + I(30) \\ 115 &= 58I \\ 58I &= 115 \\ I &= 1.98 \text{ amp. } \textit{Ans.} \end{aligned}$$

Example 2. Three resistances of 6 ohms, 8 ohms, and 10 ohms are respectively connected in parallel across a pair of line wires. The total current taken from the line is 18.8 amp. Find the line voltage and the current through each resistance.

Solution: Draw a figure illustrating the conditions of the problem. Let x, y, and z, respectively, be the currents through the 6-, 8-, and 10-ohm resistances. Indicate the assumed direction of current flow (see Fig. 48).

At point A in Fig. 48, several branches of the circuit meet. The total line current is flowing toward A and the currents in the three resistances are flowing away from A. Applying the first of Kirchhoff's laws to point A gives the equation

$$18.8 = x + y + z \qquad (1)$$

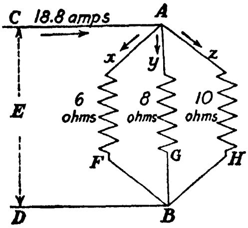


FIG. 48.

We may consider that the circuit is closed through the line voltage E, and we therefore write a voltage equation for each of the closed circuits $CAFBD$, $CAGBD$, and $CAHBD$ as follows:

$$6x = E, \qquad x = \frac{E}{6} \qquad (2)$$

$$8y = E, \qquad y = \frac{E}{8} \qquad (3)$$

$$10z = E, \qquad z = \frac{E}{10} \qquad (4)$$

Substitute these values of x, y, and z in Eq. (1), and we have

$$18.8 = \frac{E}{6} + \frac{E}{8} + \frac{E}{10} \qquad \text{The l.c.d.} = 120$$

$$18.8(120) = 20E + 15E + 12E$$

$$47E = 2{,}256$$

$$E = 48 \text{ volts.}$$

Substituting this value of E in Eqs. (2), (3), and (4) gives

$$x = 8,\ y = 6,\ z = 4.8 \text{ amp. } Ans.$$

Problems

Solve the following problems by using Kirchhoff's laws. Draw a figure for each problem:

1. Three resistances of 6, 20, and 34 ohms, respectively, are connected in series across a 120 volt generator. How much current flows?

2. A motor is connected to a generator by two line wires each of which has a resistance of 0.5 ohm. The voltage at the motor is 110 volts, and 10 amp. are flowing in the line. What is the terminal voltage of the generator?

3. Four resistances of 6, 12, 18, and 20 ohms are connected in series across a 112-volt generator. How much current flows?

4. How much resistance must be connected in series with resistances of 5.6 ohms and 8.2 ohms in series to keep the current at 5.2 amp. with 104 volts across the circuit?

5. A generator with 134.4 volts brush potential is forcing a current of 6.4 amp. through three lamps in series. If two of the lamps have

resistances of 9.3 ohms and 8.4 ohms, respectively, what is the resistance of the third lamp?

6. Two lamps A and B are in parallel across a pair of line wires. A draws 2 amp. and B draws 3 amp. Write the equation for the line current.

7. In Prob. 6, x is the line current, A takes 2 amp. and B takes y amp. Write the equation for this condition.

8. In Prob. 6, x is the line current, A takes y amp. and B takes z amp. Write the equation for this condition.

9. A circuit has three parallel branches of 6, 9, and 12 ohms, respectively. If 3 amp. flow in the 12-ohm branch, how many amperes flow in each of the others?

10. A 2- and a 3-ohm resistance are connected in parallel. The total current flowing is 12 amp. What is the e.m.f. across the resistances?

11. If the total current through the circuit of Prob. 9 were 39 amp., what would be the e.m.f. of the circuit?

12. A 3-, a 4-, and a 5-ohm resistance are connected in parallel and the sum of the currents through the three resistances is 68.5 amp. What is the e.m.f. across the resistances?

13. Repeat Prob. 12 with resistances of 4 ohms, 7 ohms, and 14 ohms. The total current is 23 amp.

14. Repeat Prob. 12 with resistances of 3 ohms, 8 ohms, and 12 ohms. The total current is 32.5 amp.

104. Three-wire Systems.—Many lighting systems and especially combined lighting and power systems are three-wire systems. The voltage between the two outside wires, in this system, is twice as great as that from either outside wire to the third wire, called the "neutral wire."

A three-wire generator is usually used to supply power to such a system, but sometimes two generators of equal voltage are connected in series with the neutral wire leading from the point where the two generators are connected together. In solving problems, we may consider the three-wire generator as being made up of two separate generators, each one having a terminal voltage half as great as the voltage between the outside wires of the three-wire machine.

Example 3. In the three-wire system illustrated in Fig. 49, each lamp draws 0.5 amp. and the e.m.f. of each generator is 112 volts. What is the voltage across each group of lamps?

Solution: Assume the generator polarities to be as indicated in Fig. 49. The current taken by group A is 6 amp. and the current taken by group

B is 7.5 amp. Assuming that the direction of these currents is as indicated, we may determine the current in the neutral wire by considering the branch point C. There are 6 amp. flowing toward this point from A and 7.5 amp. flowing away from it toward B. The current I, therefore, in the neutral wire, must be flowing toward C, and

$$I + 6 = 7.5$$
$$I = 1.5 \text{ amp.}$$

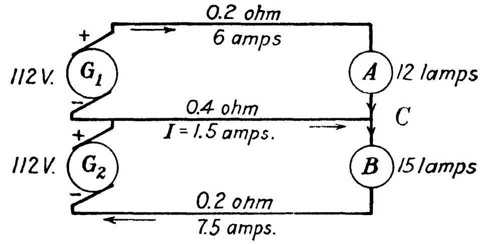


FIG. 49.

Let E_A be the e.m.f. across group A and E_B the e.m.f. across group B and write an equation for each of the closed circuits.

$$112 = 6(0.2) + E_A - 1.5(0.4) \quad (1)$$
$$112 = 1.5(0.4) + E_B + 7.5(0.2) \quad (2)$$

From Eq. (1)

$$112 - 1.2 + 0.6 = E_A$$
$$E_A = 111.4 \text{ volts } Ans.$$

From Eq. (2)

$$112 - 0.6 - 1.5 = E_B$$
$$E_B = 109.9 \text{ volts } Ans.$$

105. Current in the Neutral Wire.—The neutral wire carries the unbalanced current in a three-wire system and its value and direction can always be determined as illustrated in Ex. 3. Using the following rule, however, will simplify the matter:

Rule.—In a three-wire system, the current in any section of the neutral wire is equal to the difference of the currents in the two main wires of the same section, and its direction is the same as that of the smaller of the currents in the main wires.

Example 4. Figure 50 shows the arrangement of a three-wire system in which each lamp draws 0.75 amp. The e.m.f. of each generator is 115 volts. Find the e.m.f. across each group of lamps.

Solution: Determine the current taken by each group of lamps and mark the current and its direction in the different sections of the outside wires. Next, determine the current in each section of the neutral wire, using the rule stated in Sec. 105. Mark the direction of the currents by means of arrows, as in Fig. 50.

Let the e.m.fs across groups A, B, C, and D be, respectively, E_A, E_B, E_C, and E_D.

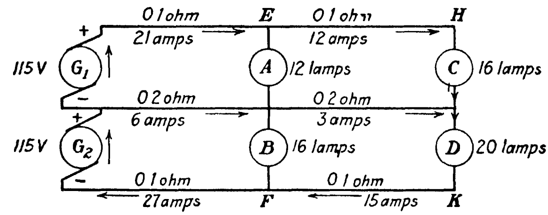


FIG. 50

Write an equation for each of the four loops in the circuit and solve them, as follows:

$$
\begin{aligned}
115 &= 21(0.1) + E_A - 6(0.2) & (1)\\
115 &= 2.1 + E_A - 1.2\\
115 - 2.1 + 1.2 &= E_A\\
E_A &= 114.1 \text{ volts}
\end{aligned}
$$

$$
\begin{aligned}
114.1 + 3(0.2) &= E_C + 12(0.1) & (2)\\
114.1 + 0.6 &= E_C + 1.2\\
114.1 + 0.6 - 1.2 &= E_C\\
E_C &= 113.5 \text{ volts}
\end{aligned}
$$

$$
\begin{aligned}
115 &= 6(0.2) + E_B + 27(0.1) & (3)\\
115 &= 1.2 + E_B + 2.7\\
115 - 1.2 - 2.7 &= E_B\\
E_B &= 111.1 \text{ volts}
\end{aligned}
$$

$$
\begin{aligned}
111.1 &= 15(0.1) + E_D + 3(0.2) & (4)\\
111.1 &= 1.5 + E_D + 0.6\\
111.1 - 1.5 - 0.6 &= E_D\\
E_D &= 109 \text{ volts}
\end{aligned}
$$

Problems

1. Draw a diagram like Fig. 49, leaving the generator voltages and line resistances unchanged. There are 6 lamps in each group and each lamp takes 0.5 amp. Find the e.m.f. across each group of lamps.

2. Repeat Prob. 1 with 12 lamps in group A and 4 lamps in group B, each lamp taking 0.5 amp.

3. Figure 50 represents an unbalanced system. To balance it, place 20 lamps in group C and 16 lamps in group D. Calculate the e.m.f. at each group of lamps with the generator voltages and line resistances unchanged and the same number of lamps in groups A and B as before. Compare these voltages with those calculated in Ex. 4.

4. Draw a diagram like Fig. 50, leaving the line resistances unchanged. Each generator has an e.m.f. of 120 volts. There are 26 lamps in group A, 24 in group B, 34 in group C, and 16 in group D. If each lamp takes 1 amp., find the e.m.f. across each group.

5. Repeat Prob. 4 with 30 lamps in group A, 40 in group B, 20 in group C, and 30 in group D. Each lamp draws 0.5 amp. and the e.m.f. of each generator is 120 volts.

6. Repeat Prob. 4 with 13 lamps in group A, 15 in group B, 27 in group C, and 32 in group D. Each lamp takes 1 amp., and the e.m.f. of each generator is 125 volts.

7. Repeat Prob. 4 with 18 lamps in group A, 20 in group B, 32 in group C, and 27 in group D. Current for each lamp and the generator voltages are as in Prob. 6.

8. In Fig. 50, suppose that group A were disconnected from the line. What would the e.m.f. across each of the other groups become?

9. In Fig. 50, calculate the e.m.f. across each of groups A, C, and D with group B disconnected from the line.

10. In Fig. 50, disconnect group C and calculate the e.m.f. across each of the other groups.

11. In Fig. 50, disconnect group D and calculate the e.m.f. across each of the other groups of lamps.

12. To the distribution system described in Prob. 4 a motor, taking 20 amp., is added and is connected across the two outside line wires at the end of the line. Find the e.m.f. across each group of lamps and across the motor.

13. In Fig. 50, a motor taking 25 amp. is added and connected to points H and K. Find the e.m.f. across each group of lamps and across the motor.

14. Connect a motor taking 20 amp. to points E and F (Fig. 50), and calculate the e.m.f. across each group of lamps and across the motor.

Example 5. If in Fig. 49 the fuse in the neutral wire is removed, calculate the resulting e.m.f. across each group of lamps, assuming that the resistance of the lamps remains unchanged.

Solution: 1. Calculate the voltage across each group of lamps with the fuse in place. From Ex. 3, these voltages are

$$E_A = 111.4 \text{ volts and } E_B = 109.9 \text{ volts}$$

2. Calculate the resistance of groups A and B under these conditions. The current through A is 6 amp. and through B it is 7.5 amp.

Let R_A = resistance of group A

And

$$R_B = \text{resistance of group } B$$

Then

$$111.4 = 6R_A$$
$$R_A = 18.57 \text{ ohms}$$

And

$$109.9 = 7.5R_B$$
$$R_B = 14.65 \text{ ohms}$$

3. With the fuse removed from the neutral wire, we have the two generators in series with the two groups of lamps and the line wires. The current in the circuit will not be the same as it was before the fuse was removed.

Let I = current after the fuse is removed

Then the voltage equation for the circuit is

$$112 + 112 = 0.2I + 18.57I + 14.65I + 0.2I$$
$$224 = 33.62I$$
$$I = 6.66 \text{ amp.}$$

$$\text{The e.m.f. across group } A = 6.66(18.57) = 123.7 \text{ volts}$$

And

$$\text{The e.m.f. across group } B = 6.66(14.65) = 97.6 \text{ volts}$$

15. In Fig. 49, with 10 lamps in group A and 16 lamps in group B each drawing 0.5 amp., what will be the e.m.f. across each group after the fuse in the neutral blows? Assume that the resistance of the lamps remains the same as before the fuse blew.

16. Repeat Prob. 15 with each lamp in group A drawing 1 amp. and each lamp in group B drawing 0.5 amp.

17. Repeat Prob. 15 with 4 lamps in group A and 10 lamps in group B. Each lamp draws 0.5 amp.

18. In Fig. 49, suppose group A consists of 24 lamps and group B of 30 lamps, each lamp taking [illegible] amp. What will be the e.m.f. across each group after the fuse in the neutral wire opens?

19. Repeat Prob. 18 with each lamp in group A drawing 1 amp. and each lamp in group B 0.7 amp.

20. Repeat Prob. 18 with each lamp in group A drawing 1 amp. and each lamp in group B 0.5 amp.

Example 6. A battery whose e.m.f. is 41.82 volts and whose internal resistance is 0.4 ohm is connected in parallel with a second battery whose e.m.f. is 37.5 volts and whose internal resistance is 1.2 ohm. These batteries supply current to a circuit having a resistance of 0.75 ohm. How much current does each battery supply and what is the current in the external circuit?

Solution: Figure 51 gives the diagram of the circuit.

Assume the directions of the currents to be as indicated by the arrows.

Let x = current through the 41.82-volt battery

And

y = current through the 37.50-volt battery

Then

$x + y$ = current through the 0.75-ohm resistance

And

$$41.82 = 0.4x + 0.75(x + y) \text{ circuit } AEFB \quad (1)$$
$$37.50 = 1.2y + 0.75(x + y) \text{ circuit } CEFD \quad (2)$$

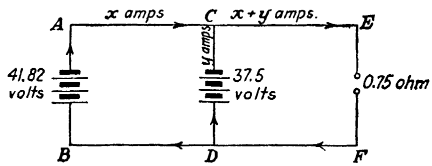


FIG. 51.

From Eq. (1),

$$41.82 = 0.4x + 0.75x + 0.75y \quad (3)$$
$$41.82 = 1.15x + 0.75y$$

From Eq. (2),

$$37.50 = 1.2y + 0.75x + 0.75y$$
$$37.50 = 0.75x + 1.95y \quad (4)$$
$$627.3 = 17.25x + 11.25y \text{ multiplying Eq. (3) by 15} \quad (5)$$
$$862.5 = 17.25x + 44.85y \text{ multiplying Eq. (4) by 23} \quad (6)$$
$$-235.2 = \quad - 33.6y \text{ subtracting Eq. (6) from Eq. (5)}$$
$$33.6y = 235.2$$
$$y = 7 \text{ amp.}$$
$$37.5 = 0.75x + 13.65 \text{ by substituting } y = 7 \text{ in Eq. (4)}$$
$$-0.75x = 13.65 - 37.5$$
$$-0.75x = -23.85$$
$$x = 31.8 \text{ amp.}$$
$$x + y = 31.8 + 7 = 38.8 \text{ amp.}$$

21. Two batteries, one having an e.m.f. of 20 volts and an internal resistance of 0.4 ohm and the other an e.m.f. of 14 volts and an internal resistance of 0.8 ohm, are connected in parallel. The external circuit has a resistance of 3.2 ohms. Find the current in each branch of the circuit and the e.m.f. across the resistance.

22. A cell whose e.m.f. is 1.5 volts, internal resistance 0.5 ohm, is connected in parallel with a cell whose e.m.f. is 1.2 volts, internal resistance 0.2 ohm. How much current flows in each branch of the circuit if a 5-ohm resistance is connected across the cells?

23. Three of the 1.5-volt cells of Prob. 22 are connected in series and the group is then connected in parallel with a group of three 1.4-volt cells joined in series. The internal resistance of each of these cells is 0.1 ohm. Find the current which this arrangement of cells will force through a 6-ohm resistance and the current supplied by each battery.

24. A cell having an e.m.f. of 1.1 volts and an internal resistance of 0.2 ohm is connected in parallel with a cell whose e.m.f. is 1.25 volts and whose internal resistance is 0.15 ohm. How much current will these cells send through an external circuit of 0.2 ohm resistance? Also, find the current through each cell and the voltage across the external circuit.

25. A 3-cell storage battery, each cell having an e.m.f. of 2.2 volts and an internal resistance of 0.01 ohm, is connected in parallel with another 3-cell battery each cell of which has an e.m.f. of 2.0 volts and an internal resistance of 0.04 ohm. Find the current in each branch of the circuit when a 0.5-ohm resistance is connected across the batteries.

26. A battery consisting of six 1.125-volt cells in series is connected in parallel with a second group of six 1.2-volt cells in series. The internal resistance for each cell of the first battery is 0.4 ohm and for each cell of the second battery it is 0.25 ohm. How much current will this combination force through an external resistance of 1.5 ohm and how much current is supplied by each battery?

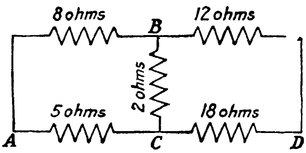


FIG. 52.

In Probs. 27 to 31 assume a battery connected across *AD*, Fig. 52.

27. In Fig. 52, the current from *A* to *C* is 10 amp. Find the current through each of the other resistances.

28. In Fig. 52, change the resistance values as follows: *A* to *B* = 4.5 ohms, *A* to *C* = 4 ohms, *B* to *C* = 1 ohm, *B* to *D* = 5 ohms, an *C* to *D* = 20 ohms. The current from *B* to *C* is 10 amp. Find the current through each of the other resistances.

29. Give the resistances in Fig. 52 the following values: *A* to *B* = 6 ohms, *A* to *C* = 10 ohms, *B* to *C* = 1[illegible] ohms, *B* to *D* = 18 ohms, and *C* to *D* = 3 ohms. The current from *C* to *D* is 8 amp. Find the current through each of the other resistances.

30. Five resistances are connected as in Fig. 52. The e.m.f. across *AD* is 23.1 volts and across *CD* it is 16.8 volts. The known resistances are: *BD* = 12 ohms, *BC* = 12 ohms, and *AC* = 7 ohms. Find the current through *BD*, through *CD*, and the value of resistances *AB* and *CD*. The current through *AB* is 1.1 amp.

31. Five resistances are connected as in Fig. 52. The current through *BD* is 6 amp. and the sum of the currents through *AB* and *AC* is 14 amp. The e.m.f. across *BC* is 12 volts. The known resistances are: *AC* = 15 ohms, *BD* = 14 ohms, and *CD* = 9 ohms. Find the current through *BC* and the value of resistances *AB* and *BC*. Battery e.m.f. is 132 volts.

CHAPTER XXIII

RATIO AND PROPORTION

106. Ratio.—The ratio of two numbers is the quotient of one of the numbers divided by the other.

107. A proportion expresses the fact that two ratios are equal. Suppose that the ratio of $a:b$ is the same as the ratio of $c:d$. We should show that this is true by writing

$$\frac{a}{b} = \frac{c}{d}$$

which is a proportion.

We know from our previous work that to clear the above equation of fractions the numerator of each fraction is multiplied by the denominator of the other fraction. This process we call "cross-multiplying," and it is the simplest way to determine whether two ratios are really equal. If the cross-products are equal, then the proportion is true.

Test each of the following statements and tell which are true proportions:

$$3/8 = 6/16;\ 8/12 = 20/30;\ 9/15 = 12/19$$
$$12/13 = 144/169;\ 8/19 = 40/95;\ 9/28 = 39/121$$

108. Variation.—If a quantity changes in value, it is said to be a variable, and if we have two quantities which are so tied together or related that if one increases or decreases in value, the other increases or decreases in the same proportion, then we have a case of direct variation, and one quantity is said to vary directly as the other.

As an illustration, take the case of the resistance of a wire as compared to its length. If we know the resistance of a certain wire, we know that a wire of the same size and material,

but ten times as long, will have a resistance ten times as great as the first wire. We say, therefore, that the resistance of a wire varies directly as its length.

If, on the other hand, we have two variables so related that if one increases the other decreases in the same proportion, then we have a case of inverse variation and one quantity is said to vary inversely as the other.

As an illustration, take the case of the resistance of a wire compared to its cross-sectional area. The resistance of a wire decreases as its cross-sectional area is increased. If we have a copper wire of given length and cross-sectional area, the resistance of a second copper wire having the same length as the first but twice its cross-sectional area will be one-half the resistance of the first wire. We say, therefore, that the resistance of a wire varies inversely as its cross-sectional area.

From the above statements, we can deduce the following relation:

$$\frac{R_1A_1}{L_1} = \frac{R_2A_2}{L_2} \tag{1}$$

where R_1, A_1, and L_1, respectively, are the resistance, cross-sectional area, and length of one wire and R_2, A_2, and L_2, respectively, are the resistance, length, and cross-sectional area of a second wire.

If the wire is circular, A is proportional to d^2, d being the diameter of the wire, and d^2 may be substituted in the above formula for A so that it becomes

$$\frac{R_1d_1^2}{L_1} = \frac{R_2d_2^2}{L_2} \tag{2}$$

109. Pulley Speeds.—It is often necessary to calculate the size of pulley necessary to obtain a given speed by belting one pulley to a second pulley which is to drive the first. If we neglect the effect of a slipping belt, it is evident that a point on the circumference of the first pulley is moving at the same rate as a point on the circumference of the second pulley. In a given time, therefore, these two points will have covered the same distance.

If D_1 and S_1 are the diameter and speed, respectively, of one pulley, D_2 and S_2 the diameter and speed, respectively, of a second pulley which is belted to the first, it follows that

$$2\pi D_1S_1 = 2\pi D_2S_2$$
$$\therefore\ D_1S_1 = D_2S_2 \tag{3}$$

Example 1. A wire 1,300 ft. long has a resistance of 0.42 ohm. What will be the resistance of 10,000 ft. of the same wire?

Solution: Since we are dealing with only one size of wire in this problem, A_1 and A_2 in Eq. (1), above, will be equal, and we may write Eq. (1) as follows:

$$\frac{R_1}{L_1} = \frac{R_2}{L_2}$$

Let x = the unknown resistance. Substitute the given values in this formula, and we have

$$\frac{0.42}{1{,}300} = \frac{x}{10{,}000}$$
$$1{,}300x = 4{,}200$$
$$x = 3.23 \text{ ohms } \textit{Ans.}$$

Example 2. Find the diameter of a wire which is 11,250 ft. long and has a resistance of 2.75 ohms, if a wire of the same material 5,500 ft. long, whose diameter is 0.65 cm., has a resistance of 4.8 ohms.

Solution: Substitute the values given in Eq. (2), Sec. 108. This gives

$$\frac{2.75d^2}{11{,}250} = \frac{4.8(0.65)^2}{5{,}500}$$
$$2.75(5{,}500)d^2 = 4.8(0.65)^2(11{,}250)$$
$$d^2 = \frac{4.8 \times 0.65 \times 0.65 \times 11{,}250}{2.75 \times 5{,}500}$$
$$d^2 = \frac{1{,}825.2}{1{,}210} \text{ by cancellation}$$
$$d^2 = 1.51$$
$$d = 1.23 \text{ cm. } \textit{Ans.}$$

Problems

In the following problems, "area" refers to the cross-sectional area of the wires:

1. A wire 1,250 ft. long has a resistance of 0.6 ohm. What will be the resistance of 1 mile of the same wire?

2. A copper wire 250 ft. long has a resistance of 0.3 ohm. How long is a copper wire of the same area whose resistance is 2.5 ohms?

3. A wire whose area is 25,000 cir. mils has a resistance of 3.5 ohms. What is the resistance of a wire of the same length and material whose area is 113,000 cir. mils?

4. A wire whose area is 25,000 cir. mils has a resistance of 3.5 ohms. What is the area of a wire of the same length and material whose resistance is 4.5 ohms?

5. What is the diameter of a wire which has a resistance of 0.015 ohm, if a wire of the same length and material, whose diameter is 0.250 in., has a resistance of 0.067 ohm?

6. What is the resistance of a wire whose diameter is 0.052 in., if a wire of the same length and material, whose resistance is 0.24 ohm, has a diameter of 0.122 in.?

7. What is the resistance of a wire whose diameter is 0.135 in., if a wire of the same length and material whose resistance is 0.17 ohm has an area of 15,500 cir. mils?

8. What is the resistance of brass wire 2,700 ft. long whose area is 137,500 cir. mils, if a brass wire 6,000 ft. long, whose area is 211,600 cir. mils, has a resistance of 0.893 ohm?

9. What is the length of a wire whose area is 27,500 cir. mils and whose resistance is 2.5 ohms, if a wire of the same material, 6,250 ft. long and with an area of 129,500 cir. mils, has a resistance of 2.0 ohms?

10. A wire whose area is 250,000 cir. mils is 3,500 ft. long and has a resistance of 1.1 ohms. What would be the area of a wire of the same material which is 575 ft. long and has a resistance of 0.35 ohm?

11. A wire 1,200 ft. long, 0.345 in. in diameter, has a resistance of 0.25 ohm. Find the resistance of a wire of the same material which is 3,570 ft. long and 0.275 in. in diameter.

12. A wire 2,750 ft. long, 0.85 cm. in diameter, has a resistance of 0.375 ohm. Find the resistance of a wire of the same material which is 6,500 ft. long and 0.65 cm. in diameter.

13. What is the length of a wire 1.05 cm. in diameter whose resistance is 1.75 ohms, if a wire of the same material, 3,600 ft. long and 2.35 cm. in diameter, has a resistance of 2.05 ohms.

14. 3,750 ft. of wire, 1.55 cm. in diameter, have a resistance of 6.5 ohms. How long is a wire of the same material which has a resistance of 4.5 ohms and a diameter of 1.3 cm.?

15. Find the area of a wire which is 12,000 ft. long and has a resistance of 7.5 ohms, if a wire of the same material, 3,950 ft. long having an area of 350,000 cir. mils, has a resistance of 1.2 ohms.

16. 8,500 ft. of wire, having an area of 175,000 cir. mils, have a resistance of 2.6 ohms. What is the area of a wire which has a resistance of 4.75 ohms and is 10,500 ft. long?

17. 18,500 ft. of wire, 0.325 in. in diameter, have a resistance of 3.9 ohms. What is the diameter of a wire which has a resistance of 1.25 ohms and is 11,000 ft. long?

18. Find the diameter of a wire which is 12,000 ft. long and has a resistance of 3.6 ohms if a wire 4,600 ft. long, having a diameter of 0.55 cm., has a resistance of 5.8 ohms.

19. Find the diameter of a wire which is 9,900 ft. long and has a resistance of 2.2 ohms if a wire 12,100 ft. long, having a diameter of 0.627 in., has a resistance of 1.69 ohms.

20. How long is a wire which has a resistance of 27 ohms and a diameter of 0.09 in., if a wire 8,100 ft. long, having a diameter of 0.39 in., has a resistance of 4.5 ohms?

21. Find the resistance of a wire 27,500 ft. long and 1.2 cm. in diameter if a wire 8,700 ft. long, 0.92 cm. in diameter, has a resistance of 5.8 ohms.

22. A motor whose armature rotates at 1,500 r.p.m. has a 5-in. pulley on its armature shaft. What should be the diameter of the generator pulley if the generator is to rotate at 1,200 r.p.m.?

23. Find the speed at which the armature of a motor rotates if it is belted to a generator having a 6.5-in. pulley rotating at 1,600 r.p.m. The motor pulley has a diameter of 5.5 in.

CHAPTER XXIV

SIMILAR FIGURES. TRIGONOMETRIC FUNCTIONS

110. Similar figures are figures which are identical in shape but usually not of the same size. These figures have the characteristic that corresponding parts are in proportion. For example, of similar triangles we know that the corresponding sides are in proportion and that the corresponding angles are equal.

The triangles in Fig. 53 are similar. The corresponding sides of the triangles are, therefore, in proportion, and we may write the proportions

$$\frac{BA}{DE} = \frac{BC}{DF} \text{ and } \frac{BC}{DF} = \frac{AC}{EF}$$

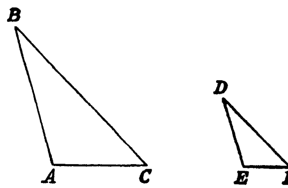


Fig. 53.—Two similar triangles.

Other proportions can be obtained from the same triangles, such as

$$\frac{AC}{AB} = \frac{EF}{ED}, \quad \frac{DF}{DE} = \frac{BC}{BA}, \text{ etc.}$$

The properties of similar triangles are used extensively in surveying, in determining the heights of inaccessible objects, the distance across a swamp, or some other obstruction. Similar triangles also form the basis of trigonometry.

111. Conditions Determining Similarity.—If two triangles are similar, the corresponding angles of the triangles are equal. If two angles of a triangle are equal to two angles of another triangle, the third angle of one must also be equal to the third angle of the other. If two angles of a triangle are equal to two angles of a second triangle, therefore, we know that the two triangles are similar.

In right triangles, the right angles are equal. If, therefore, we have two right triangles in which an acute angle of one is

equal to an acute angle of the other, the two triangles are similar.

In any triangle, if the corresponding sides are in proportion, the triangles are similar.

Example 1. A building is 69.5 ft. high. What is the length of its shadow at a time when a yard-stick, held upright with one end touching the ground, casts a shadow which is 2 ft. 9 in. long?

Solution: The conditions of this example may be illustrated by two similar right triangles, as shown in Fig. 54, in which AB represents the

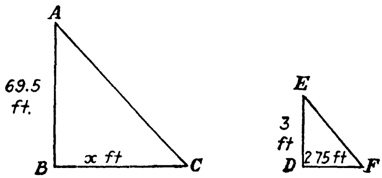


FIG. 54.

height of the building and BC the length of its shadow, while ED represents the yardstick and DF its shadow.

From the two similar triangles in Fig. 54 we obtain the proportion

$$\frac{69.5}{x} = \frac{3}{2.75}$$

$$3x = 2.75(69.5)$$

$$3x = 191.125$$

$$x = 63.7 \text{ ft. } Ans.$$

Example 2. In the right triangle (Fig. 55*a*), $CK = 20$, $KB = 15$, and $AB = 30$ ft. Find the length of HK.

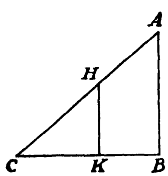


FIG. 55*a*.

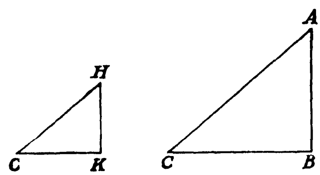


FIG. 55*b*.

Solution: Separate the triangle shown in Fig. 55*a* into two figures, as shown in Fig. 55*b*.

Since $CK = 20$ and $KB = 15$ ft., $CB = 20 + 15 = 35$ ft.
From the triangles of Fig. 55*b* we obtain the proportion

$$\frac{CK}{HK} = \frac{CB}{AB}$$

$$\frac{20}{x} = \frac{35}{30} \text{ by substituting the known values}$$

$$35x = 600$$

$$x = 17.1 \text{ ft. } Ans.$$

Problems

1. Two triangles are similar. The sides of the first are 6 in., 10 in., and 14 in. The longest side of the second triangle is 39 in. Find the length of the other two sides.

2. Two triangles have equal angles. The sides of the first measure 7 ft., 8.5 ft., and 12 ft. The shortest side of the second measures 10.5 ft. How long are the other two sides of the second triangle?

3. If you were asked to make a drawing of a triangular-shaped lot whose sides measure 100 ft., 85 ft., and 67 ft., how long would each side in your drawing be if you used a scale of $1\frac{1}{2}$ in. = 10 ft.?

4. A boy finds that the shadow of a 6-ft. pole measures 5 ft. at the same time that the shadow cast by a telephone pole measures 37.5 ft. How far above the ground is the top of the telephone pole?

5. Draw a right triangle ABC with AB the hypotenuse. Draw a line from a point D on the hypotenuse parallel to AC. This line meets the other side at E. If AC measures 9 in., DE 5 in., and EB 8 in., find the length of CB.

6. Using the figure for Prob. 5, suppose E represents a point on the bank of a river and B is a tree on the opposite bank. DE and AC are laid off parallel to the river bank. AC measures 90 ft., ED 75 ft., and CE 30 ft. Find the width of the river.

7. In the figure for Prob. 5, suppose $AC = 39$ ft., $CE = 12$ ft., and $EB = 40$ ft. Find DE.

8. Suppose in Prob. 5 that $AD = 16$ ft., $BD = 44$ ft., and $EB = 40$ ft. Find CB.

9. Suppose in Prob. 5 that $DE = 46$, $AC = 69$, and $EB = 95$. Find CE.

10. A chimney is 150 ft. high. How long is its shadow at a time when a 4-ft. rod casts a shadow 4 ft. 6 in. long?

11. How high is the Woolworth building if it casts a shadow 1,100 ft. long at the same time that the Singer building, which is 612 ft. high, casts a shadow 850 ft. long?

12. What will be the length of the shadow of the Metropolitan Life building, which is 700 ft. high, when the Singer building casts a shadow 918 ft. long?

112. Special Types of Triangles.—We can conceive of a triangle in which two sides are of the same length or one in which the three sides are of the same length. These are special figures.

An isosceles triangle is a triangle two of whose sides are of the same length.

An equilateral triangle is a triangle whose three sides are of the same length.

In an isosceles triangle, two of the angles are equal. These angles are always opposite the equal sides.

In an equilateral triangle, the three angles are equal.

Oral Exercises

13. Two angles of a triangle are 67 and 24°. How large is the third angle?

14. Two angles of a triangle are acute. What kind of angle is the third?

15. One angle of a triangle is obtuse. What kind of angles are the other two?

16. Can there be two obtuse angles in one triangle?

17. How many degrees are there in the two acute angles of a right triangle?

18. One angle of a triangle has 90°. The other two angles are equal. How many degrees in each angle? What kind of triangle is it?

19. The three sides of a triangle are equal. How many degrees in each angle? What kind of triangle is it?

20. One angle of a right triangle is 30°. How many degrees in the other acute angle?

21. One of the base angles of an isosceles triangle is 36°. How large is the other base angle?

22. A perpendicular is drawn from the vertex of the equal sides of an isosceles triangle. If the base is 4 in. long, how long is each of the parts into which the base is divided?

23. One side of an equilateral triangle is 6 in. A perpendicular is drawn from one vertex to the side opposite. Into what parts is this side divided?

24. What are similar triangles? State the relation that exists between corresponding sides of similar triangles.

25. What relation exists between the corresponding angles of similar triangles?

Written Exercises

26. In a triangle, the lengths of the sides are 6 in., 10 in., and 15 in. If the longest side of a similar triangle measures 25 in., how long are the other two sides?

27. In the triangle ABC, C is a right angle. A line is drawn perpendicular to AC from a point E on AB. This line meets AC at D. If $AD = 6$ in., $DC = 2$ in. and $AB = 12$ in., find the length of AE.

28. In Prob. 27, if $AE = 8$ in., $EB = 3$ in., and $AC = 9$ in., find AD and DC.

29. In Prob. 27, if $AD = 9$ and $DC = 3$, find DE if $BC = 18$.

30. In Prob. 29, find BC if $DE = 18$.

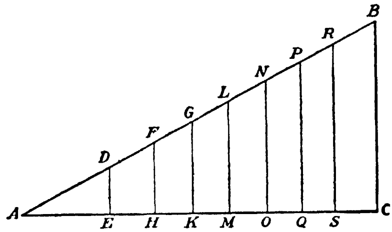


FIG. 56.

31. Write at least 12 equal ratios from the similar triangles in Fig. 56.

32. Draw a right triangle ABC, with C the right angle. Which side is opposite angle A? Which side is opposite angle B? Which side is adjacent to angle A? Which side is adjacent to angle B? Which side is the hypotenuse?

33. If one acute angle of a right triangle is equal to an acute angle of a second right triangle, are the triangles similar? Will the ratios of corresponding sides be equal?

113. Functions of an Angle.—It will be seen from Fig. 56 that if we draw any acute angle A and drop perpendiculars from points on one side of the angle to the other side, definite equal ratios exist between the sides of the right triangles so formed. That is, if we choose any pair of sides of one triangle, the ratio of every other pair of corresponding sides of the other triangles will be the same as the ratio of the first two sides. Note that the ratio has the same value no matter what the length of the sides may be.

If the angle A is made larger or smaller, these sets of equal ratios will change in value, but for any given size of A, they will always be the same. Because of this fact, these ratios

have been named and their values have been determined for different sizes of angles. These ratios are called "functions of an angle"; we shall define and use three of these functions.

In the right triangle ABC, AB is the hypotenuse, BC is the side opposite the angle A, and AC is the side adjacent to the angle A.

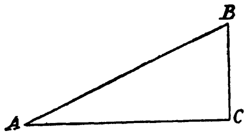


FIG. 57.

The *sine* of an angle is the ratio of the side opposite the angle to the hypotenuse.

The *cosine* of an angle is the ratio of the side adjacent to the angle to the hypotenuse.

The *tangent* of an angle is the ratio of the side opposite the angle to the side adjacent to the angle.

The above definitions are based on the ratios of the sides of any *right triangle.*

For the angle A, in Fig. 57, these ratios are

$$\text{Sine } A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{BC}{AB}$$

$$\text{Cosine } A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{AC}{AB}$$

$$\text{Tangent } A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{BC}{AC}$$

For the angle B, in Fig. 57, these ratios are

$$\text{Sine } B = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{AC}{AB}$$

$$\text{Cosine } B = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{BC}{AB}$$

$$\text{Tangent } B = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{AC}{BC}$$

The sine, cosine, and tangent are abbreviated sin, cos, and tan.

A *function* is a variable whose value depends upon the value of some other quantity or quantities.

The sine, cosine, and tangent are functions of an angle, because their value depends upon the size of the angle.

114. Use of Trigonometric Functions.—In solving alternating-current problems, it is necessary that the student know what the sine, cosine, and tangent of an angle are and he must know how to use them. The following problems are designed to give the necessary practice in the use of these functions:

In each of the following problems, determine to 4 decimal places the value of the sine, cosine, and tangent of angles A and B. The letters refer to Fig. 57. Draw a figure to illustrate each problem.

1. $AB = 30$	$AC = 24$	$BC = 18$
2. $AB = 78$	$AC = 30$	$BC = 72$
3. $AB = 25$	$AC = 24$	$BC = 7$
4. $AB = 17$	$AC = 15$	$BC = 8$
5. $AB = 26$	$AC = 10$	$BC = 24$
6. $AB = 61$	$AC = 11$	$BC = 60$
7. $AB = 85$	$AC = 84$	$BC = 13$
8. $AB = 41$	$AC = 9$	$BC = 40$
9. $AB = 29$	$AC = 21$	$BC = 20$
10. $AB = 113$	$AC = 112$	$BC = 15$

115. Use of the Table of Natural Functions.—In the appendix, you will find a table giving the natural values of the sines, cosines, and tangents of angles from 0 to 90 deg. You will note that the first page of the table (p. 194, Appendix) is marked "0°" and "1°" at the top and "88°" and "89°" at the bottom. This can be done because each value in the table is a function of two different angles. For example, the sine of 57 deg. is the same as the cosine of 33 deg., the cosine of 22 deg. has the same value as the sine of 68 deg., etc. To find the sine, cosine, or tangent for angles of less than 45 deg., we read from the top of the page downward, and to find the functions of angles greater than 45 deg. we read upward from the bottom of the page, as explained in detail in the examples which follow.

Example 3. Find sin 14° 25′.

Solution: Turn to the page of the table which is marked "14°" at the top. Follow downward the column marked minutes (′) until you reach 25′. Opposite 25, in the column marked "N Sin" at the top, read 0.24897.

Note that this value 0.24897 is also the cosine of 75° 35′.

Example 4. Find cos 64° 37′.

Solution: On the page marked "64°," follow the right-hand column of minutes upward to 37′. Opposite 37, in the column marked "N Cos" at the bottom, read 0.42867.

Example 5. tan A = 0.65646. Find A.

Solution: Turn to any page of the table of natural functions and read the first value from the top of the column marked "N Tan" at the top of the page and, also, read the first value from the bottom of the column marked "N Tan" at the bottom of the page. If the value sought lies between these two values and is not on the page you are looking at, it will be found on a page farther along in the table. If the value sought does not lie between the two values read from the table, it will be found on one of the earlier pages. Continue until you have found the number 0.65646 on the page marked "33°" opposite to 17 in the minute column.

$$\therefore A = 33° \ 17'$$

Example 6. cos B = 0.55605. Find B.

Solution: Proceed as in Ex. 5. The number 0.55605 will be found in a column marked "N Cos" at the bottom of the page. We must, therefore, read the number of degrees from the bottom of the page and the number of minutes from the right-hand column of minutes.

$$\therefore B = 56° \ 13'$$

Problems

From the table of natural functions find the following:

1. sin 37° 54′	**3.** cos 65° 48′	**5.** sin 85° 51′
2. tan 39° 27′	**4.** tan 72° 36′	**6.** cos 21° 16′

Find the angles corresponding to the following values:

7. sin A = 0.36921	**9.** cos B = 0.89101	**11.** cos A = 0.37973
8. tan A = 0.89777	**10.** tan C = 2.3220	**12.** sin X = 0.80902

116. Angle of Elevation, Angle of Depression.—The angle of elevation or depression at any point is the angle between a horizontal line drawn through the point of observation and the line connecting this point with the object under observation. In Fig. 58, angle CAB is the angle of elevation of the point B at the point A, and angle DBE is the angle of depression of the point E at the point B. Also, angle DBE is equal to angle BEC.

Example 7. The angle of elevation of the top of a cliff, standing in a level plain, is 36° 15′ at point A located 150 ft. from the base of the cliff. Find the height of the cliff above the plain.

Solution: Figure 59 illustrates the conditions of the problem, BC representing the cliff.

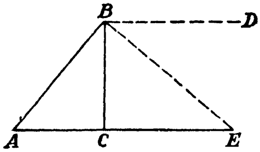


FIG. 58.—Illustration of angles of elevation and depression.

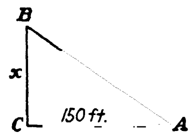


FIG. 59.

Angle A is given, and the given side AC is adjacent to angle A, while the required side BC is opposite angle A. We therefore use the tangent function, which deals with the opposite and the adjacent sides.

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\tan 36^\circ 15' = \frac{x}{150}$$

$$\text{But } \tan 36^\circ 15' = 0.73323$$

$$\therefore 0.73323 = \frac{x}{150}$$

$$x = 0.73323(150)$$

$$x = 109.98 = 110.0 \text{ ft. } Ans.$$

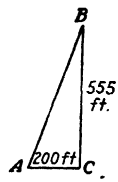


FIG. 60.

Example 8. The Washington monument is 555 ft. high. Find to the nearest minute the angle of elevation of its top at a point 200 ft. from the base of the monument.

Solution: The two given sides are opposite and adjacent to the required angle. Therefore, we use

$$\tan A = 555/200$$

$$\tan A = 2.7750$$

$$\therefore A = 70^\circ 11' \ Ans.$$

The student will experience little difficulty in determining which function to use in the solution of problems if he observes the following rules:

Rule 1.—*When an angle and a side of the triangle are given, take the given side and the required side and use that function of the given angle which involves these two sides.*

Rule 2.—*When two sides are given, use that function of the required angle which involves the two given sides.*

Problems

1. At the base of a mountain, the angle of elevation of its top is observed to be 48° 50′. The straight-line distance from this point to the top of the mountain is 12,000 ft. How high is the mountain?

2. The angle of elevation of the top of a factory chimney is 36° 12′ at point *A*, located 200 ft. from the base of the chimney. Find the height of the chimney.

3. A stairway rises 2 ft. for every 3 ft. measured along the stairway. Find the angle of elevation of the stairway.

4. A buoy is located 300 ft. from the base of a lighthouse. If the angle of depression of the buoy at the top of the lighthouse is 14° 38′. find the distance from the top of the lighthouse to the buoy.

5. An observation balloon is 300 ft. above point *A*, located on the ground directly beneath the balloon. An observer in the balloon notes that the angle of depression of a distant point *B* is 1° 9′. Find the distance from *A* to *B*.

6. Two observers *A* and *B* are directly in line with and on the same side of a tower, which is 135 ft. high. *A* measures the angle of elevation of the top of the tower to be 26° 12′, and *B* finds that it is 35 37′. How far apart are *A* and *B*?

7. A tower which is 160 ft. high is directly between two points *A* and *B*. The angle of elevation of the top of the tower is 40° 22′ at *A*, and at *B* it is 60° 32′. Find the distance from *A* to *B*.

8. Calculate the angle of elevation of an airplane at point *A* located 2 miles from *B*, if the airplane is 2,200 ft. above the ground and directly over *B*.

9. At the top of a lighthouse, the angle of depression of a boat is observed to be 30° 46′. The distance from the top of the lighthouse to the boat is 275 ft. Find the height of the lighthouse.

10. How far away from the base of a spire is an observer who notes that the angle of elevation of the top of the spire is 22° 36′? The top of the spire is 120 ft. above the ground.

117. Interpolation.—It is often necessary to find the value of the function of an angle, whose size is given, in degrees, minutes, and seconds. Since the table of natural functions gives the values to degrees and minutes only, we must estimate how large a change will be caused in the value of the desired function by the given number of seconds. This process is called *interpolation.* It is explained in the two examples which follow.

Example 9. Find the value of sin 31° 13′ 11″.

Solution: From the table, we find sin 31° 14′ = 0.51852

And sin 31° 13′ = 0.51828

Subtracting, the difference for 1′ = 0.00024

Neglecting decimals, the difference for 60″ = 24

The difference for 1″ = $^{24}/_{60}$

And the difference for 11″ = $\frac{24 \times 11}{60} = 4.4 = 4$

Since the value of the sin increases as the angle increases, this difference for 11″ must be added to the value of sin 31° 13′.

sin 31° 13′ = 0.51828

The difference for 11″ = 0.00004

sin 31° 13′ 11″ = 0.51832 *Ans.*

Example 10. Find the value of cos 24° 38′ 42″.

Solution.

cos 24° 38′ = 0.90899

cos 24° 39′ = 0.90887

The difference for 60″ = 0.00012

The difference for 42″ = $\frac{12 \times 42}{60} = 8.4 = 8$

Since the value of the cosine decreases as the angle increases, this difference must be subtracted from the value of cos 24° 38′.

cos 24° 38′ = 0.90899

The difference for 42″ = 0.00008

cos 24° 38′ 42″ = 0.90891 *Ans.*

Example 11. tan A = 0.87425. Find angle A.

Solution: Turn to the table of natural functions and find the two values of tangents which are closest to the given value. In this case, we find 0.87389 and 0.87441 which are the values for angles 41° 9′ and 41° 10′, respectively.

tan 41° 10′ = 0.87441

tan 41° 9′ = 0.87389

The difference for 1′ = 0.00052

We know that the desired angle lies between the two values 41° 9′ and 41° 10′, and we can represent it by 41° 9′ x″.

tan 41° 9′ x″ = 0.87425

tan 41° 9′ = 0.87389

The difference for x″ = 0.00036

And the difference for 60″ = 0.00052

$$\frac{x}{60} = \frac{0.00036}{0.00052}$$

$$\frac{x}{60} = \frac{36}{52}$$

$$52x = 2{,}160$$

$$x = 41.54'' = 42''$$

$$\therefore A = 41^\circ\ 9'\ 42''\ \textit{Ans.}$$

Problems

Find the value of the sine, cosine, and tangent of the following angles:

1. 10° 42′ 16″
2. 21° 32′ 24″
3. 27° 16′ 8″
4. 35° 22′ 51″
5. 42° 12′ 38″
6. 48° 27′ 5″
7. 53° 54′ 58″
8. 59° 21′ 43″
9. 31° 48′ 27″
10. 15° 13′ 32″

Find the value of angle A in each of the following:

11. $\sin A = 0.37294$
12. $\cos A = 0.37294$
13. $\tan A = 0.37294$
14. $\cos A = 0.76495$
15. $\tan A = 1.2648$
16. $\sin A = 0.84291$
17. $\cos A = 0.27642$
18. $\tan A = 2.6451$
19. $\sin A = 0.38426$
20. $\sin A = 0.87651$
21. $\cos A = 0.24721$
22. $\sin A = 0.02645$
23. $\tan A = 0.03241$
24. $\tan A = 0\ 02127$
25. $\sin A = 0.54726$
26. $\tan A = 1.5473$

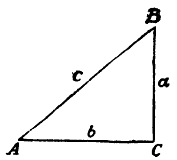


Fig. 61.—Method of lettering a right triangle.

118. The standard method of lettering a right triangle is shown in Fig. 61.

119. Solution of Right Triangles.—To solve a triangle requires that we find the value of the unknown sides and angles of the triangle.

Example 12. In a right triangle, $A = 22° 36' 12''$ and $a = 17.65$. Solve the triangle.

Solution: Using the given side and the given angle, we can determine the length of sides b and c.

$$\tan A = \frac{a}{b} \qquad \sin A = \frac{a}{c}$$

$$\tan 22° 36' 12'' = 0.41633 \qquad \sin 22° 36' 12'' = 0.38435$$

$$\therefore 0.41633 = \frac{17.65}{b} \qquad \therefore 0.38435 = \frac{17.65}{c}$$

$$0.41633b = 17.65 \qquad 0.38435c = 17.65$$

$$b = 42.39 \qquad c = 45.92$$

$$A + B = 90° \quad \therefore B = 90° - A$$

$$\begin{aligned} 90° &= 89° 59' 60'' \\ A &= 22° 36' 12'' \\ \hline \therefore B &= 67° 23' 48'' \end{aligned}$$

Problems

Solve the following right triangles:

1. Given $A = 21°$, $a = 20$
2. Given $B = 36°$, $c = 42$
3. Given $A = 38° \, 27'$, $c = 50$
4. Given $A = 49° \, 45'$, $b = 40$
5. Given $a = 18$, $c = 36$
6. Given $a = 25$, $b = 14$
7. Given $b = 65$, $c = 75$
8. Given $B = 36° \, 29' \, 16''$, $c = 12$
9. Given $A = 58° \, 36' \, 22''$, $c = 44$
10. Given $a = 125$, $b = 176$
11. Given $b = 220$, $c = 277$
12. Given $A = 72° \, 54' \, 48''$, $b = 112$
13. Given $B = 27° \, 31' \, 35''$, $b = 210$
14. Given $a = 219$, $c = 312$
15. Given $a = 230$, $b = 93$
16. Given $B = 78° \, 49' \, 30''$, $b = 215$
17. Given $b = 450$, $c = 1120$
18. Given $A = 15° \, 37' \, 45''$, $c = 500$
19. Given $A = 19° \, 41' \, 30''$, $b = 250$
20. Given $a = 215$, $c = 916$

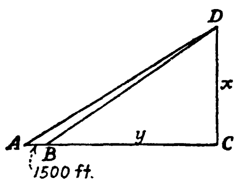


FIG. 62.

Example 13. A man wishes to determine the height of a distant mountain. At point A, he measures the angle of elevation of the top of the mountain to be $31° \, 23'$. At B, which is 1,500 ft. closer to the mountain than A, the angle of elevation of the top of the mountain is $34° \, 27'$. Calculate the height of the mountain.

Solution:

$$\tan A = \frac{x}{AC}$$
$$\tan A = 0.61000$$
$$\text{and } AC = 1{,}500 + y$$
$$\therefore 0.61 = \frac{x}{1{,}500 + y}$$
$$x = 0.61(1500 + y)$$
$$x = 915 + 0.61y \qquad (1)$$

$$\tan B = \frac{x}{y}$$
$$\tan B = 0.68600$$
$$\therefore 0.686 = \frac{x}{y}$$
$$x = 0.686y \qquad (2)$$

Substitute the value of x from Eq. (2) in Eq. (1), and we have

$$0.686y = 915 + 0.61y$$
$$0.686y - 0.61y = 915$$
$$0.076y = 915$$
$$y = 12{,}039$$
$$x = 0.686(12{,}039) = 8{,}259 \text{ ft. } \textit{Ans.}$$

Problems

1. A man observes that the angle of elevation of the top of a monument in a level plain is $47° \, 59'$. Keeping in line with the monument and the first point, he walks 400 ft. toward the monument, where he finds that the angle of elevation is $71° \, 20'$. Find the height of the monument.

2. In Prob. 1, suppose that the elevation at the first point is 36° 44′ and at the second point 49° 22′. How far apart are these two points, if the monument is 750 ft. high?

3. From the top of a lighthouse, the angles of depression of two boats in line with the lighthouse are 21° 32′ and 48° 53′, respectively. If the top of the lighthouse is 130 ft. above the water level, how far apart are the two boats?

4. Two points A and B are in line with the top of a mountain which rises in a level plain. The angles of elevation of the mountain top are 12° 31′ at A and 21° 54′ at B. The distance from A to B is 7 miles. Find the height of the mountain above the plain.

5. In Prob. 4, if the angles of elevation are 21° 46′ at A and 50° 26′ at B, what is the distance between A and B if the top of the mountain is 1,000 ft. above the level plain?

6. If the cosine of an angle is equal to 0.70, what is the sine of the same angle?

7. $\sin A = 0.75$, find $\cos A$.

8. Find the height of a tree which has an angle of elevation of 36° 19′ 27″ when measured at a point 100 ft. from the base of the tree.

9. A building is 590 ft. high. Find to the nearest second the angle of elevation of its top measured at a point on the ground which is 250 ft. from the base of the building.

10. $\tan B = 0.90$, find $\cos B$.

11. $\tan x = 0.62$, find $\sin x$.

12. $\cos y = 0.45$, find $\sin y$.

CHAPTER XXV

FUNDAMENTAL ALTERNATING-CURRENT IDEAS

120. Alternating Current.—An alternating current is one which periodically reverses its direction of flow. Thus far, we have dealt only with steady direct currents. The study of alternating currents involves many new factors which have no effect when the current is steady and unidirectional. In this chapter, we shall consider only some of the elementary facts concerning alternating currents.

121. Instantaneous Value.—An alternating current does not have a steady value for any appreciable period of time, but its value is constantly changing and each instant the current has a different value than it had the preceding instant. The e.m.f. which causes the current flow varies in the same manner. If we take the value of the current or e.m.f. at any one instant, we have what is known as an *instantaneous* value of current or e.m.f.

122. Alternation. Cycle.—The variations in the e.m.f. and current in an alternating-current circuit are exceedingly rapid. For a short period of time, all of the successive instantaneous values act in one direction. If during this period of time the initial values of current and e.m.f. are zero, the instantaneous values of current and e.m.f. will then increase until each reaches a maximum value from which point they will begin to decrease until both again reach zero. Such a set of instantaneous values from zero to a maximum and back to zero constitute one *alternation.* Any two successive alternations comprise one *cycle.*

123. Electrical Degree.—A cycle may be divided into 360 equal periods of time each of which represents one *electrical degree.*

124. Sine Curve.—When a conductor is rotated at uniform speed through a uniform magnetic field, the instantaneous

values of e.m.f. induced in the conductor are proportional to the sine of the angle through which the conductor has been moved from its neutral position. A curve which shows the variations in the instantaneous value of such an e.m.f. during a complete cycle is called a *sine curve.*

Figure 63 represents a sine curve of e.m.f. The horizontal line is divided into electrical degrees and the vertical distance from any point on the curve to the horizontal line represents the instantaneous value of the e.m.f. at that point in the cycle.

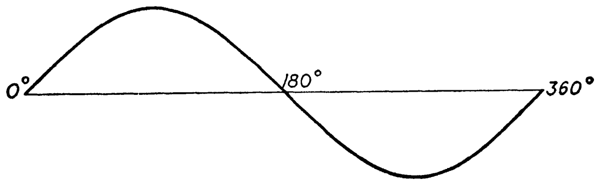


FIG. 63.—A sine curve.

125. Maximum Value.—When a conductor has been rotated 90 deg. from its neutral position, the e.m.f. induced in the conductor has reached its largest instantaneous value, and this value is known as the "maximum value" of the e.m.f. If we consider the instantaneous values during the first half of the cycle as positive and those during the second half as negative, then the positive maximum is reached at the 90-deg. position in the cycle and the negative maximum is reached at the 270-deg. position. The neutral position of a conductor, when the instantaneous values are about to increase in the direction taken as positive, is usually considered to be the starting point of a cycle and is known as the "0-deg." position.

126. The formula for the instantaneous value of an e.m.f. in terms of the sine of the angle through which the conductor has been rotated from its neutral position and the maximum e.m.f. is

$$\sin \alpha = \frac{e}{E_{\max}}$$

where α is the angle through which the conductor has been moved from its neutral position

e is the instantaneous value of the e.m.f.

$E_{\max}$ is the maximum value of the e.m.f.

Similarly,

$$\sin \alpha = \frac{i}{I_{max}}$$

where i is the instantaneous value of current

I_{max} is the maximum value of current

By referring to Fig. 64, we shall see that the above formula is a reasonable one, although the following is not intended as a proof of the formula. In the figure, P represents a conductor which is being moved at a uniform speed through a uniform magnetic field in the path represented by the circle. The instantaneous value of e.m.f. for any position of the conductor is represented by the vertical distance from the circumference of the circle to the horizontal line marked "0°" and "180°." Hence, AP represents the instantaneous value at point P, and OP represents the maximum value, since it is the radius of the circle and it is evident that at the 90-deg. position the instantaneous value is represented by the radius of the circle.

Therefore, in the right triangle OAP,

$$\sin \alpha = \frac{AP}{OP} = \frac{e}{E_{max}}$$

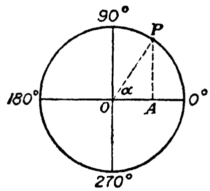


FIG. 64.

Example 1. Find the instantaneous value of an e.m.f. when 61° 32′ of its cycle have been completed. The maximum value of the e.m.f. is 1,200 volts.

Solution:

$$\sin \alpha = \frac{e}{E_{max}}$$

$$\sin 61° \, 32' = \frac{e}{1{,}200}$$

$$0.87909 = \frac{e}{1{,}200}$$

$$e = 0.87909(1{,}200)$$

$$e = 1{,}054.9 = 1{,}055 \text{ volts } \textit{Ans.}$$

127. The sine of an angle between 90 and 180 deg. is found by taking the sine of 180 deg. minus the angle. Thus, if x is an angle which is larger than 90 and smaller than 180 deg.

$$\sin x = \sin(180° - x)$$

Example 2. Find the maximum value of an alternating current whose instantaneous value is 25.1 amp. 36° 11′ after the plus maximum value has been reached.

Solution: The plus maximum value occurs at 90°; therefore

$$\alpha = 90° + 36°\ 11' = 126°\ 11'$$

$$\text{And } \sin \alpha = \sin 126°\ 11' = \sin (180° - 126°\ 11') = \sin 53°\ 49'$$

$$\sin \alpha = \frac{i}{I_{\max}}$$

$$\sin 53°\ 49' = \frac{25.1}{I_{\max}}$$

$$0.80713 = \frac{25.1}{I_{\max}}$$

$$0.80713 I_{\max} = 25.1$$

$$I_{\max} = \frac{25.1}{0.80713}$$

$$I_{\max} = 31.1 \text{ amp. } \textit{Ans.}$$

128. The sine of an angle between 180 and 270 deg. is found from the relation

$$\sin y = -\sin(y - 180°)$$

where y is an angle larger than 180 and smaller than 270 deg.

Example 3. Find the instantaneous value of an alternating e.m.f. when 243° 11′ of its cycle have been completed. The maximum value of the e.m.f. is 600 volts.

Solution: $\sin \alpha = -\sin (243°\ 11' - 180°) = -\sin 63°\ 11'$

$$\sin \alpha = \frac{e}{E_{\max}}$$

$$-\sin 63°\ 11' = \frac{e}{600}$$

$$-0.89245 = \frac{e}{600}$$

$$e = 0.89245(600)$$

$$e = -535.47 = -535.5 \text{ volts } \textit{Ans.}$$

129. The sine of an angle between 270 and 360 deg. is determined by using the relation

$$\sin z = -\sin(360° - z)$$

where z is an angle larger than 270 and smaller than 360 deg.

Example 4. An alternating current has a maximum value of 60 amp. How many degrees of its cycle has it completed when its value is -44 and its value is decreasing?

Solution:

$$\sin \alpha = \frac{-44}{60}$$

$$\sin \alpha = -0.73333$$

$$\alpha = 47° \ 10'$$

Since the given instantaneous current occurs in the last quarter of the cycle, the number of degrees of the cycle which have been completed will be between 270 and 360 deg. For angles between 270 and 360 deg.

$$\sin z = -\sin (360° - z)$$

But -0.73333 is the sine of an angle between 270 and 360 deg.

$$\therefore 47° \ 10' = 360° - z$$

$$z = 360° - 47° \ 10'$$

$$z = 312° \ 50' \ \textit{Ans.}$$

Problems

1. An alternating e.m.f. has a maximum value of 800 volts. Find the instantaneous values of this e.m.f. at the following positions in the cycle: 15°, 24° 30′, 30°, 35° 45′, 45°, 60°, 75° 30′, and 85° 20′.

2. An alternating current has a maximum value of 60 amp. Find its instantaneous values at the following positions in the cycle: 20° 45′, 75° 50′, 100° 30′, 170° 40′, 185°, 255° 10′, 300°, and 330° 25′.

3. An alternating e.m.f. has a value of 100 volts when 48° 30′ of its cycle have been completed. What is its maximum value?

4. An alternating current has a value of 38.2 amp. when 61° 45′ of its cycle have been completed. Find the maximum value of this current.

5. The instantaneous value of an alternating e.m.f. when 36° 13′ of its cycle are completed is 267 volts. Find its instantaneous values when (*a*) 80° 17′ of the cycle have been completed, (*b*) 121° 49′ of the cycle have been completed.

6. An alternating current has a value of 16.4 amp. when 25° 13′ of its cycle have been completed. How many degrees and minutes of the cycle will have been completed when its value is (*a*) +28 and decreasing, (*b*) −28 and increasing, (*c*) −28 and decreasing?

7. The instantaneous value of an alternating e.m.f. is 145 volts when 12° 13′ of its cycle are complete. What will be its value at the following positions in the cycle: (*a*) 75° 13′, (*b*) 162° 17′, (*c*) 218° 39′, (*d*) 298° 46′?

8. Find the instantaneous value of an alternating current (*a*) 68° 28′ after its maximum value of +38.7 amp., (*b*) 25° 13′ before its negative maximum, (*c*) 36° 48′ after its negative maximum, (*d*) 21° 13′ after the 180° position.

9. An alternating e.m.f. has a value of 1,215 volts 71° 30′ after its plus maximum value. What is its value at (*a*) 212° 45′, (*b*) 320° 17′?

10. When an alternating e.m.f. has completed 17° 38′ of its cycle, it has a value of 165 volts. Find its values at the following positions in the cycle: (*a*) 76° 19′, (*b*) 148° 17′, (*c*) 236° 19′.

130. The effective value of an e.m.f. is the square root of the average square of the instantaneous values which make up its cycle. The effective e.m.f. is represented by the symbol E_{eff}, and its relation to the maximum e.m.f. is expressed by the formula

$$E_{eff} = \frac{E_{max}}{\sqrt{2}}$$

This is usually written $E_{eff} = 0.707E_{max}$

Similarly, $I_{eff} = 0.707I_{max}$

Problems

1. Find the effective value of an e.m.f. whose maximum value is 1,100 volts.

2. Find the effective value of a current whose maximum value is 75 amp.

3. The effective value of an alternating current is 50 amp. What is its maximum value?

4. The effective value of an alternating e.m.f. is 440 volts. Find its maximum value.

5. The effective value of an alternating e.m.f. is 220 volts. What is its value when it has completed 110° 25′ of its cycle?

6. The effective value of an alternating current is 30 amp. What is its value when it has completed 74° 30′ of its cycle?

7. An alternating e.m.f. has an instantaneous value of 200 volts at the 35° position in its cycle. What is the effective value of this e.m.f.?

8. What is the effective value of an alternating current which has a value of 26.5 amp. when it has completed 130° 30′ of its cycle?

9. The effective value of an alternating current is 40 amp. How many degrees and minutes of its cycle has it completed when its value is 25 amp. and is decreasing?

10. The effective value of an alternating e.m.f. is 1,100 volts. How many degrees and minutes of its cycle has it completed when its value is −800 volts and is decreasing?

131. Impedance.—The impedance of a circuit may be defined as *the total opposition to the flow of current in the circuit.* When the circuit under consideration carries a constant direct current, the total opposition to the flow of current is the physical property of the conducting materials known as *resistance.* When an alternating e.m.f. is applied to a circuit, there is, in addition to the resistance, a second factor known as *reactance.* The combined effect of these two factors is known as the *impedance* of the circuit. Resistance and reactance cannot be added directly but must be considered as though they were two forces directed at right angles to each other. Since this is true, we may illustrate the relation between resistance, reactance, and impedance by means of a right triangle, as follows:

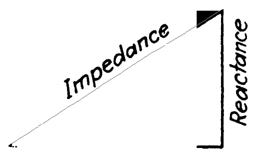


FIG. 65.—The impedance triangle.

The conventional symbols for these quantities are Z = impedance, R = resistance, and X = reactance. Since these quantities may be related to the sides of a right triangle, it is evident that

$$Z^2 = R^2 + X^2$$

Ohm's Law.—The general statement of Ohm's law, often referred to as "Ohm's law for alternating-current circuits," is: *the e.m.f. of a circuit is equal to the current, expressed in amperes, multiplied by the impedance, expressed in ohms.* This statement, in symbols, is

$$E = IZ$$

In this formula E and I represent the effective values of voltage and current, respectively, and Z represents the impedance in ohms.

***Example* 5.** A circuit has a resistance of 5 ohms and a reactance of 12 ohms at 60 cycles. How much current will pass through this circuit if the impressed e.m.f. is 120 volts, 60 cycles?

Solution: 1. Find the impedance of the circuit, using the formula

$$Z^2 = R^2 + X^2$$
$$Z^2 = (5)^2 + (12)^2$$
$$Z^2 = 25 + 144$$
$$Z^2 = 169$$
$$Z = 13 \text{ ohms}$$

2. Find the current, using Ohm's law.

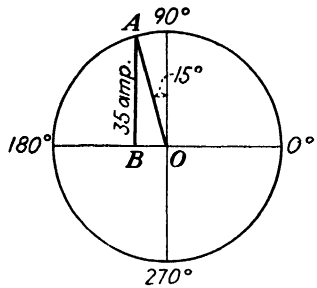


FIG. 66.

$$E = IZ$$
$$120 = I(13)$$
$$I = 120 \div 13 = 9.23 \text{ amp. } \textit{Ans.}$$

Example 6. The instantaneous value of the current in a coil 15° after its plus maximum value, is 35 amp. What would be the reading of a voltmeter connected across the coil, if the impedance of the coil is 5.5 ohms?

Solution: 1. Find the maximum value of current.

Since the plus maximum value occurs at 90°, AB, in Fig. 66, represents the current 15° later.

$$\text{Angle } OAB = 75°$$
$$\therefore \sin 75° = \frac{35}{I_{\text{max}}}$$
$$0.9659 = \frac{35}{I_{\text{max}}}$$
$$I_{\text{max}} = 35 \div 0.9659 = 36.2$$

2. Find the effective current

$$I = 0.707 I_{\text{max}}$$
$$I = 0.707(36.2) = 27.87 \text{ amp.}$$

3. Find the reading of the voltmeter.

$$E = IZ$$
$$E = 27.87(5.5) = 153 \text{ volts } \textit{Ans.}$$

Voltmeters and ammeters read effective values of voltage and current. Since 27.87 is the effective current, 153 is the effective voltage and is, therefore, the reading of the voltmeter.

Problems

1. What is the impedance of a coil which has a resistance of 50 ohms and a reactance of 40 ohms?

2. The impedance of a coil is 100 ohms. If its inductive reactance is 50 ohms, what is the resistance of the coil?

3. A circuit has an impedance of 45 ohms. The current flowing is 1.3 amp. What is the e.m.f. across the circuit?

4. Find the impedance of a coil which draws 4.6 amp. when connected across a 110-volt 25-cycle line.

5. The resistance of a coil is 45 ohms. Find the reactance of the coil if its impedance is 84 ohms.

6. A coil whose impedance is 13 ohms is connected across a 110-volt 60-cycle line. How much current flows? What is the maximum value of this current?

7. A circuit has an impedance of 35.2 ohms. What is the effective voltage if the maximum value of the current is 11.4 amp?

8. The e.m.f. across an impedance coil has a maximum value of 59.7 volts. An ammeter in series with the coil indicates a current of 9.6 amp. What is the coil impedance?

9. What is the impedance of a circuit which has a resistance of 48 ohms and a capacitive reactance of 64 ohms?

10. A noninductive resistance is connected in series with a condenser. The impedance of the combination is 79 ohms, while the condenser has a reactance of 52 ohms. What is the value of the resistance?

11. The 60-cycle impedance of a circuit is 37 ohms. When connected across a 60-cycle source of supply the current in the circuit is 1.06 amp. at the instant when 18° of the cycle have been completed. If a voltmeter is connected across this circuit, what should its reading be?

12. The instantaneous value of the current in a certain circuit is 15 amp. when 40° of its cycle have been completed. If the impedance of the circuit is 12 ohms, how many volts would be indicated by a voltmeter connected across the circuit?

13. A voltmeter connected across a coil whose impedance is 14.5 ohms reads 225 volts. Find the current value at the instant when 40° of the current cycle have been completed.

14. An e.m.f. has a value of 120 volts when 29° of its cycle have been completed. What would be the effective value of current if this e.m.f. were connected across a circuit of 36.4 ohms impedance?

15. What is the reactance of a condenser which is connected in series with a noninductive 49-ohm resistance, if the impedance of the combination is 98 ohms?

16. The e.m.f. across a coil is 220 volts. Find the current if the coil resistance is 39 ohms and its reactance is 46 ohms.

17. In a certain circuit the current is 36.2 amp. at the instant when 47° of the current cycle have passed. The maximum value of the circuit voltage is 247.5 volts. Find the circuit impedance.

18. A coil whose impedance is 15.8 ohms is connected across a line where the e.m.f. is 200 volts. Find the instantaneous value of current 29° after its zero value.

19. The instantaneous value of current in a circuit is 3.75 amp. when 39° of the current cycle have been completed. Find the impedance of the circuit, if a voltmeter which is connected across the circuit reads 104 volts.

20. A circuit whose resistance is 68 ohms carries 2.25 amp. when the impressed e.m.f. is 220 volts, 60 cycles. What is the reactance of this circuit at 60 cycles?

21. Find the resistance of a coil which draws 3 amp. from a 220-volt 25-cycle line. The coil reactance is 34 ohms at 25 cycles.

22. An ammeter indicates 13.5 amp. when connected in a certain circuit. The impedance of the circuit is 12.5 ohms. What is the maximum value of the voltage supply?

23. A circuit has an impedance of 37.8 ohms. What would a voltmeter read when connected across this circuit, if the instantaneous current is 4.9 amp. 75° after the 0° position?

24. The instantaneous current value, 20° after zero, is 12.8 amp. A voltmeter when connected across the circuit indicates 219 volts. What is the impedance of the circuit?

25. What effective voltage is necessary to force 11.5 amp. through a circuit whose resistance is 9.1 ohms and whose reactance is 7.2 ohms?

26. The instantaneous value of a current 13° after its plus maximum value is 38.6 amp. Find the impedance of the circuit if the impressed e.m.f. is 115 volts.

27. The impedance of a circuit is 12.8 ohms. Twenty-five degrees after its plus maximum value the instantaneous value of the e.m.f. is 88.7 volts. Calculate the instantaneous value of current 65° after its zero value.

28. The resistance of a circuit is 24 ohms. The impedance of the circuit is twice as large as its reactance. Calculate the impedance and the reactance of the circuit.

29. The reactance of a circuit is equal to $\frac{1}{3}$ of its impedance. If the resistance of the circuit is 24 ohms, calculate the impedance and reactance values.

30. A certain circuit has a resistance of 18 ohms, and its impedance is 2.6 times as large as its reactance. If the impressed e.m.f. is 130 volts, how much current will pass through the circuit?

132. Power.—In an alternating-current circuit containing reactance, the true power used is not equal to the product of the effective values of voltage and current. This means that the direct-current formula $W = EI$ does not hold good for alternating-current circuits. This product EI is called the *apparent power* or the *volt-amperes* of the circuit. The abbreviation va. is used for volt-amperes. Apparent power

is often more conveniently expressed in kilovolt-amperes (kva.), which is the number of volt-amperes divided by one thousand. The *true power* or *effective power* absorbed by a circuit is equal to the product of EI and the cosine of the angle between the current and the voltage. The Greek letter θ is used to represent the angle between the current and the voltage. It is called the *angle of lead* when the current leads the voltage, which it does when the reactance is capacitive, and it is called the *angle of lag* when the current lags behind the voltage, which it does when the reactance is inductive. The *reactive power* is the power absorbed by the reactance of a circuit during one period of the cycle and is returned to the circuit during the period which follows immediately. Cos θ is called the *power factor* because it is the ratio of the effective power to the apparent power, or, in other words, it is the decimal factor by which the apparent power must be multiplied in order to obtain the true power. Figure 67 shows the relation between these quantities.

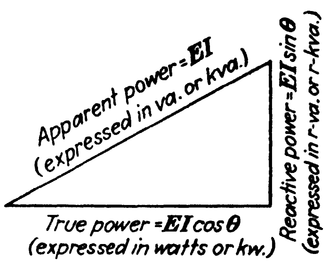


FIG. 67.—The power triangle.

The facts outlined in the above are expressed by the following formulas:

$$(\text{Apparent power})^2 = (\text{true power})^2 + (\text{reactive power})^2 \quad (1)$$

$$\text{Power factor} = \cos\theta = \frac{\text{true power}}{\text{apparent power}} \quad (2)$$

$$\cos\theta = \frac{R}{Z} \quad (3)$$

$$\sin\theta = \frac{X}{Z} \quad (4)$$

$$W = I^2R \quad (5)$$

$$W = EI\cos\theta \quad (6)$$

Example 7. A coil has a reactance of 10 ohms at 60 cycles. The power factor of the coil is 76 per cent. How much power will this coil absorb when connected across a 220-volt 60-cycle line?

Solution: 1. Find the impedance of the coil, using (4).

From the power-factor table* we note that when the power factor or $\cos\theta$ is 0.76, then $\sin\theta$ is 0.65.

$$\sin\theta = \frac{X}{Z}$$
$$\therefore 0.65 = \frac{10}{Z}$$
$$0.65Z = 10$$
$$Z = 10 \div 0.65 = 15.38 \text{ ohms}$$

2. Find the current through the coil using Ohm's law.

$$E = IZ$$
$$220 = I(15.38)$$
$$I = 220 \div 15.38 = 14.3 \text{ amp.}$$

3. Find the power absorbed, using (6).

$$W = EI\cos\theta$$
$$W = 220(14.3)(0.76)$$
$$W = 2{,}391 \text{ watts } \textit{Ans.}$$

Example 8. The apparent power absorbed by a circuit is 1.3 kva., and the reactive power absorbed is 0.5 r-kva. (*a*) What is the true power used by the circuit? (*b*) What is the power factor? (*c*) What is the resistance of the circuit if the current is 25 amp.?

Solution: 1. Find the true power.

$$(\text{Apparent power})^2 = (\text{true power})^2 + (\text{reactive power})^2$$
$$(1.3)^2 = x^2 + (0.5)^2$$
$$1.69 = x^2 + 0.25$$
$$1.69 - 0.25 = x^2$$
$$1.44 = x^2$$
$$1.2 = x$$
$$x = 1.2 \text{ kw. } \textit{Ans.}$$

2. Find the power factor.

$$\text{Power factor} = \frac{\text{true power}}{\text{apparent power}}$$
$$\text{P.f.} = \frac{1.2}{1.3} = 0.923 \text{ or } 92.3 \text{ per cent } \textit{Ans.}$$

3. Find the resistance of the circuit.

$$W = I^2R$$
$$1{,}200 = (25)^2R$$
$$1{,}200 = 625R$$
$$625R = 1{,}200$$
$$R = 1.92 \text{ ohms } \textit{Ans.}$$

* Page 249.

Example 9. A coil whose impedance is 30 ohms and whose power factor is 65 per cent, takes 5 kw. when connected across a supply line. How much current is taken by the coil and what is the line voltage?

Solution: 1. Find the resistance of the coil.

$$\cos\theta = \frac{R}{Z}$$
$$0.65 = \frac{R}{30}$$
$$30(0.65) = R$$
$$R = 19.5 \text{ ohms}$$

2. Find the current.

$$W = I^2R$$
$$5{,}000 = I^2(19.5)$$
$$I^2 = 5{,}000 \div 19.5$$
$$I^2 = 256.4$$
$$I = \sqrt{256.4} = 16 \text{ amp. } \textit{Ans.}$$

3. Find the line voltage.

$$W = EI\cos\theta$$
$$5{,}000 = E(16)(0.65)$$
$$5{,}000 = 10.4E$$
$$E = 5{,}000 \div 10.4 = 481 \text{ volts } \textit{Ans.}$$

Problems

1. The e.m.f. across a circuit is 110 volts, and the current in the circuit is 12 amp. The angular difference between current and voltage is 12°. How much power is consumed in the circuit?

2. A wattmeter connected in a circuit reads 1,526 watts. The e.m.f. of the circuit is 220 volts, and the current is 7.5 amp. Calculate the power factor of the circuit.

3. The current in a circuit is 25.7 amp., and the e.m.f. is 225 volts. If the power factor is 78 per cent, what should be the reading of a wattmeter which is connected so that it reads the power absorbed by the circuit?

4. The impedance of a circuit is 18 ohms, and its power factor is 75 per cent. Calculate the resistance and reactance of the circuit.

5. A coil whose power factor is 80 per cent has a resistance of 22 ohms. Calculate the impedance and reactance of the coil.

6. A circuit uses 3.25 kw. at 82 per cent power factor and 220 volts pressure. What is the current in the circuit?

7. The e.m.f. across a coil is 220 volts. Calculate the current through the coil if the coil absorbs 4.8 kw. at a power factor of 75 per cent.

8. A wattmeter connected in a circuit reads 2,980 watts. The e.m.f. impressed on the circuit is 440 volts, and the current in the circuit is 8.2 amp. What is the power factor of the circuit?

9. A circuit which absorbs 460 watts draws 6.8 amp. at a power factor of 60 per cent. Calculate the e.m.f. of the circuit.

10. An e.m.f. of 225 volts is impressed on a circuit whose impedance is 25.7 ohms. How much power is used by this circuit if its power factor is 82 per cent?

11. A coil has a resistance of 6.5 ohms and uses 936 watts. Calculate the current through the coil.

12. A motor draws 25 amp. from a 115-volt line. What is its power factor, if the power used is 2,500 watts?

13. A circuit which has an impedance of 32.6 ohms draws 1.2 kw. from a 215-volt alternator. What is the power factor of the circuit?

14. In a certain circuit the true power is 4.5 kw., and the reactive power is 2.9 *r*-kva. Calculate the apparent power.

15. A noninductive resistance is connected in series with a condenser whose reactance is 7.41 ohms. The power factor of the combination is 90 per cent. Find the impedance and the resistance of this combination. How much current flows if the impressed e.m.f. is 115 volts?

16. 115 volts are impressed on a coil whose reactance and power factor at the frequency of the impressed e.m.f. are 1.62 ohms and 70 per cent, respectively. Calculate the impedance and resistance of the coil and the power used.

17. The apparent power in a circuit is 5.9 kva. The true power used by the circuit is 4,200 watts. Calculate the reactive power.

18. A coil whose impedance is 28.9 ohms is connected across a 110-volt circuit. If the coil uses 300 watts under these conditions, what is its resistance? What is the reactance of the coil?

19. A 115-volt e.m.f. is impressed on a circuit whose impedance is 11.5 ohms. What is the apparent power used? If the circuit absorbs 950 watts, calculate the resistance of the circuit and its power factor.

20. An e.m.f. of 220 volts is impressed on a coil whose resistance is 10.8 ohms and whose power factor is 60 per cent. Find the impedance and reactance of the coil and the power used.

21. A circuit has an impedance of 22 ohms and a power factor of 65 per cent. Calculate the resistance and reactance of the circuit.

22. A coil having an impedance of 19 ohms and a reactance of 15.2 ohms at 60 cycles is connected across a 110-volt 60-cycle line. Calculate the power absorbed by the coil under these conditions.

23. A coil whose resistance is 32.4 ohms has a reactance of 13.5 at 25 cycles. How much power does it absorb when it is connected to 115-volt 25-cycle mains?

24. A circuit whose impedance is 18.5 ohms takes a lagging current at a power factor of 87 per cent. What are the resistance and reactance of the circuit under these conditions?

25. A circuit whose 60-cycle reactance is 5 ohms draws 12 amp. when the impressed e.m.f. is 156 volts, 60 cycles. Calculate the power used by the circuit.

26. An appliance whose resistance is 24.5 ohms has a power factor of 75 per cent at 25 cycles. Find the 25-cycle impedance and reactance of this appliance.

27. An appliance uses 1.4 kw. at 88 per cent power factor when connected to a 110-volt line. How much current does it draw?

28. An appliance takes 7 kw. at 80 per cent power factor from a pair of line wires. Under these conditions the impedance of the appliance is 27.3 ohms. Calculate the current through the appliance and the line voltage.

29. If the e.m.f. of a circuit remains constant but the resistance is doubled and the power is cut in half, what change occurs in the current? In the reactance?

30. If the e.m.f. of a circuit remains constant but the impedance is doubled and the power factor is four-fifths of its original value, what change is made in the resistance? In the power?

133. Inductance is that property of an electric circuit which enables it to build up an e.m.f. by electromagnetic induction whenever the current strength in the circuit changes. This induced e.m.f. is always in such a direction that it opposes the change in current and, hence, retards the current change. The unit of inductance is the *henry*, which is defined as that inductance which will induce an e.m.f. of one volt when the current in the circuit is changing at the rate of one ampere per second.

134. Capacitance is that property of a condenser which enables it to retain an electric charge which opposes any changes in the voltage of the circuit in which the condenser is connected. The unit of capacitance is the *farad.* This, however, is a very large unit so that, in practice, capacitance is usually given in terms of the microfarad. One million microfarads (mf.) are equal to one farad.

135. Reactance is the effect of inductance or capacitance expressed in ohms, and the two types of reactance resulting from these effects are termed "inductive reactance" and "capacitive reactance," respectively. The formula for each follows:

$$X_L = 2\pi f L$$

$$X_c = \frac{10^6}{2\pi f C}$$

In the above formulas X_L is the inductive reactance, X_c is the capacitive reactance, L is the inductance in henrys, C is the capacitance in microfarads, and f is the frequency in cycles per second. π equals 3.1416.

It is convenient to consider the quantity $2\pi f$ as a constant at any given frequency. The values of $2\pi f$ for frequencies ranging from 10 cycles to 100 cycles are given in the table below:

f	$2\pi f$	f	$2\pi f$	f	$2\pi f$
10	62.8	40	251	70	440
15	94.2	45	283	75	471
20	126	50	314	80	503
25	157	55	346	85	534
30	188	60	377	90	565
35	220	65	408	95	596

Example 10. Find the impedance of a coil at a frequency of 100 cycles if the coil resistance is 12 ohms and its inductance is 0.036 henry.

Solution: 1. Find the reactance of the coil.

$$X_L = 2\pi f L$$
$$X_L = 628(0.036) = 22.6 \text{ ohms}$$

2. Find the impedance of the coil.

$$Z^2 = R^2 + X^2$$
$$Z^2 = (12)^2 + (22.6)^2$$
$$Z^2 = 144 + 510.76$$
$$Z = \sqrt{654.76} = 25.6 \text{ ohms } Ans.$$

Example 11. What is the capacitance of a condenser which draws 5 amp. when connected across a 120-volt 60-cycle line?

Solution: 1. Determine the reactance of the condenser.

$$E = IZ$$
$$120 = 5Z$$
$$Z = 24 \text{ ohms} = X_c \text{ (since } R = 0)$$

2. Determine the capacitance of the condenser.

$$X_c = \frac{10^6}{2\pi f C}$$
$$24 = \frac{10^6}{377C}$$
$$24(377C) = 10^6$$
$$C = \frac{10^6}{24(377)} = 110 \text{ mf. } Ans.$$

From the formulas given under Sec. 135 it is apparent that the reactance of a coil having a fixed inductance varies directly as the frequency, and that the reactance of a condenser having a fixed capacitance varies inversely as the frequency. When the reactance at a certain frequency is known and it is desired to obtain the reactance at a second frequency, application of these principles, as expressed by the following formulas, will save time.

$$\frac{X_{L1}}{X_{L2}} = \frac{f_1}{f_2}$$

$$\frac{X_{c1}}{X_{c2}} = \frac{f_2}{f_1}$$

In these formulas X_{L1} and X_{c1} are, respectively, the inductive and capacitive reactances at frequency f_1; X_{L2} and X_{c2} are, respectively, the inductive and capacitive reactances at frequency f_2.

Example 12. A coil which has a resistance of 2.5 ohms takes 10 amp. from 220-volt 60-cycle mains. How much current will it take from 110-volt 25-cycle mains?

Solution: 1. Calculate the 60-cycle impedance of the coil.

$$\begin{aligned} E &= IZ \\ 220 &= 10Z \\ Z &= 220 \div 10 = 22 \text{ ohms (at 60 cycles)} \end{aligned}$$

2. Calculate the 60-cycle reactance of the coil.

$$\begin{aligned} Z^2 &= R^2 + X^2 \\ (22)^2 &= (2.5)^2 + X^2 \\ 484 &= 6.25 + X^2 \\ 484 - 6.25 &= X^2 \\ 477.75 &= X^2 \\ 21.88 &= X \text{ (at 60 cycles)} \end{aligned}$$

3. Calculate the 25-cycle reactance of the coil.

$$\begin{aligned} \frac{X_{L1}}{X_{L2}} &= \frac{f_1}{f_2} \\ \frac{21.88}{X_{L2}} &= \frac{60}{25} \\ 60X_{L2} &= 21.88(25) \\ X_{L2} &= \frac{21.88(25)}{60} \\ X_{L2} &= 9.12 \text{ ohms (reactance at 25 cycles)} \end{aligned}$$

4. Calculate the 25-cycle impedance of the coil.

$$\begin{aligned} Z^2 &= R^2 + X^2 \\ Z^2 &= (2.5)^2 + (9.12)^2 \\ Z^2 &= 6.25 + 83.1744 \\ Z^2 &= 89.4244 \\ Z &= 9.46 \text{ ohms (at 25 cycles)} \end{aligned}$$

5. Calculate the current.

$$E = IZ$$
$$110 = I(9.46)$$
$$I = 110 \div 9.46 = 11.6 \text{ amp. } \textit{Ans.}$$

Problems

1. A condenser has a capacitance of 2 mf. Calculate its reactance at 25, 60, and at 1,000 cycles.

2. A coil has an inductance of 0.05 henry. Calculate its reactance at 25, 60, and at 1,000 cycles.

3. Find the reactance of a 0.005-mf. condenser at 60 and at 1,000,000 cycles.

4. Find the 25- and the 60-cycle reactance of a coil whose inductance is 0.25 henry.

5. The inductive reactance of a coil at 60 cycles is 2.6 ohms. Calculate its reactance at 25, 40, and 150 cycles.

6. The reactance of a condenser is 4.5 ohms at 60 cycles. Calculate its reactance at 25, 40, and 150 cycles.

7. A coil has a resistance of 6 ohms and an inductance of 0.021 henry. How much current will this coil draw (*a*) from a 50-volt 60-cycle line, and (*b*) from a 50-volt 100-cycle line?

8. A coil has a resistance of 10 ohms and an inductance of 0.015 henry. How much current will this coil draw from (*a*) a 60-cycle 75-volt line, and from (*b*) a 100-cycle 75-volt line?

9. How much current will be taken from a 50-volt line at 25 and at 40 cycles by a coil which has a resistance of 27 ohms and an inductance of 0.18 henry?

10. An impedance coil has a resistance of 10 ohms and a reactance of 6 ohms at 60 cycles. What will be the impedance of this coil at 25, 100, and 250 cycles?

11. An impedance unit consists of a noninductive resistance of 12 ohms in series with a condenser whose reactance at 60 cycles is 5 ohms. Calculate the impedance of this unit at 25, 40, and 150 cycles.

12. A noninductive resistance of 7 ohms is connected in series with a 300-mf. condenser. Find the impedance of this combination, the current taken, and the power absorbed when the impressed e.m.f. is (*a*) 120 volts, 25 cycles; (*b*) 120 volts, 60 cycles; and (*c*) 120 volts, 1,000 cycles.

13. A condenser takes 5.5 amp. from 220-volt 25-cycle mains. How much current will it draw from 220-volt 40-cycle mains? From 220-volt 60-cycle mains?

14. A circuit consists of a noninductive resistance of 25 ohms in series with a 40-mf. condenser. What is the 60-cycle impedance of this circuit? How much current and how much power will be taken by this circuit if the impressed e.m.f. is 220 volts, 60 cycles?

15. Find the impedance of a 2-mf. condenser connected in series with a 3,000-ohm noninductive resistance at 60 cycles. How much current will this combination take from a 110-volt 60-cycle line?

16. An impedance coil draws 15 amp. from a 225-volt 60-cycle line. Its 60-cycle reactance is 12 ohms. Find (*a*) its impedance at 60 cycles; (*b*) its resistance; (*c*) its reactance at 25 cycles; (*d*) its impedance at 25 cycles; and (*e*) the current which it will draw when connected across a 110-volt 25-cycle line.

17. An impedance coil whose resistance is 12 ohms draws 7 amp. from a 110-volt 25-cycle line. How much current will it draw from 220-volt 60-cycle mains?

18. A coil has an impedance of 38 ohms and a power factor of 75 per cent at 25 cycles. What is its impedance at 60 cycles?

19. A circuit consists of a condenser in series with a 25-ohm noninductive resistance. Its impedance at 25 cycles is 50 ohms. Calculate its impedance at 60 cycles.

20. At what frequency will the reactance of a 10-mf. condenser be 63.66 ohms?

21. A coil whose resistance is 12.8 ohms draws 7 amp. from a 112-volt 60-cycle line. What is the inductance of the coil?

22. An inductive appliance has an impedance of 25 ohms at 25 cycles and a power factor of 80 per cent. How much current will this appliance take from a 220-volt 60-cycle line?

23. A coil has a resistance of 40 ohms and an inductance of 0.05 henry. What is its power factor at a frequency of 60 cycles?

24. A coil whose impedance is 12.5 at 60 cycles is connected across a 120-volt 60-cycle line. Find the resistance of the coil if 1,000 watts is being used. What are the apparent power used, the power factor, and the inductance of the coil?

25. An impedance unit consisting of a noninductive resistance in series with a condenser draws 25 amp. from a 225-volt 60-cycle circuit. The 60-cycle reactance of the condenser is 8.3 ohms. Find for this unit (*a*) its impedance at 60 cycles, (*b*) its resistance, (*c*) its reactance at 150 cycles, (*d*) its impedance at 150 cycles, and (*e*) the current which it will draw from 110-volt 150-cycle mains.

26. An impedance unit consisting of a 15-ohm noninductive resistance in series with a condenser draws 13 amp. from a 221-volt 25-cycle line. How much current will this unit take from 112-volt 40-cycle mains?

27. The impedance unit described in Prob. 26 will take what current from 220-volt 60-cycle mains?

28. A coil whose impedance at 60 cycles is 16.8 ohms takes 756 watts when connected across 126-volt 60-cycle mains. Find the apparent power used, the power factor, the resistance of the coil, and the inductance of the coil.

29. A coil whose resistance is 8 ohms draws 6 amp. when connected across a certain 110-volt line. What is the frequency of this line, the inductance of the coil being 0.0263 henry?

30. A coil has a resistance of 6 ohms and an inductance of 0.12 henry. What is the current taken by this coil when the impressed e.m.f. is 110

volts, 60 cycles? How large a resistance must be connected in series with the coil when the impressed e.m.f. is 110 volts direct current in order that the current will have the same value?

31. A coil takes 20 amp. when connected to 120-volt 60-cycle mains. When connected across a 6.2-volt storage battery the current is 12.4 amp. Find the resistance and inductance of the coil.

32. A coil takes 20 amp. when the impressed e.m.f. is 220 volts, 25 cycles. This coil is connected in series with a noninductive resistance of 4 ohms across 50-volt d.-c. mains and the current is then 6.25 amp. Calculate the resistance and inductance values of the coil.

33. A noninductive resistance is connected in series with a condenser across 120-volt 60-cycle mains. Under these conditions 480 watts are used at a power factor of 40 per cent. Calculate the value of the resistance and the capacitance of the condenser.

34. A coil whose impedance is 24 ohms absorbs 364.5 watts when connected across a 60-cycle 108-volt e.m.f. Calculate the apparent power, the power factor, the resistance of the coil, and the inductance of the coil.

35. A circuit consisting of a condenser in series with a noninductive resistance is connected across a 220-volt 25-cycle line. The current in the circuit is 10 amp., and the e.m.f. across the resistance is 150 volts. Calculate the resistance and capacitance.

36. A noninductive resistance of 12 ohms is connected in series with a 300-mf. condenser across a 220-volt line. The power used is 1,930 watts. Calculate the frequency, apparent power, and power factor.

37. An impedance coil has a reactance of 12 ohms at 25 cycles and takes 8 amp. from a 440-volt 25-cycle line. What current will this coil draw when connected across a 110-volt 60-cycle line?

38. A coil whose resistance is 20 ohms draws 4.4 amp. from a 110-volt 25-cycle line. How much current will it draw from a 110-volt 60-cycle line?

39. A coil has a resistance of 8 ohms and an inductance of 0.2 henry. How much current will this coil draw when connected across a 220-volt 25-cycle line? How much resistance must be added to the coil so that the same amount of current will be drawn when the e.m.f. is changed to 220 volts direct current?

40. A coil takes 20 amp. from a 60-volt 60-cycle line. When connected across a 6-volt storage battery the current is 4 amp. What are the resistance and inductance values?

41. A coil takes 15 amp. from 120-volt 60-cycle mains. When this coil is connected in series with a 7-ohm noninductive resistance across a 50-volt d.-c. source, the current is 5 amp. Calculate the resistance and inductance of the coil.

CHAPTER XXVI

VECTORS. COMPLEX QUANTITIES

136. A line whose length, direction, and position accurately represents a given quantity is a *vector*, and the quantity so represented is a *vector quantity*. Electrical quantities, such as current, voltage, power, etc., are vector quantities and must be treated as such when we are dealing with alternating-current circuits.

137. The complex-quantity method of representing vectors is generally used to designate electrical quantities which must be treated as vector quantities. This system is illustrated by Fig. 68.

In Fig. 68, XX' and YY' are any two straight lines inter-

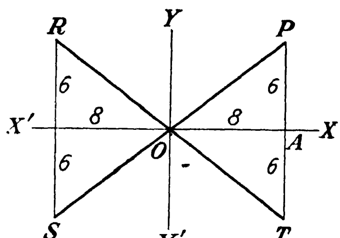


FIG. 68.—Vector designation by complex quantities.

secting at right angles at point O. This point is called the *origin* and OX is known as the *initial line* or the *reference line*. Distances measured in the directions OX and OY are positive; distances measured in the directions OX' and OY' are negative. Suppose OP to be a vector in the first quadrant whose length and position are to be accurately described. We may do this by drawing PA perpendicular to OX and then describing OP in terms of OA and AP. In other words, OP can be described in terms of its horizontal component OA and its vertical com-

ponent AP. If $OA = 8$ and $AP = 6$, we may completely describe OP, as to length and position, by saying that its horizontal component is 8 and its vertical component is 6. In order to do this conveniently, the operator j is used in the complex-quantity method. The symbol $+j$ indicates a change in direction of 90° and that the indicated vertical component is measured in the positive direction, which is upward from XX'; the symbol $-j$ indicates a change in direction of 90° and that the indicated vertical component is measured in the negative direction, or downward from XX'.

To represent the vector OP all we need do, therefore, is to write

$$\overline{OP} = 8 + j6$$

which means that to reach point P from O we move a distance of eight units in the positive direction, change our direction by 90° and then move a distance of six units in the positive direction to reach point P. The vinculum over OP is used to distinguish the vector quantity $\overline{OP}$ from its numerical or scalar value OP.

Similarly, the vectors in the remaining quadrants are represented as follows:

$$\begin{aligned} \overline{OR} &= -8 + j6 \\ \overline{OS} &= -8 - j6 \\ \overline{OT} &= 8 - j6 \end{aligned}$$

From the complex expression of a vector we can determine its numerical or scalar value and the sine, cosine, or tangent of the angle between the vector and OX. For example,

$$\begin{aligned} \overline{OT} &= 8 - j6 \\ (OT)^2 &= (8)^2 + (6)^2 \\ (OT)^2 &= 100 \\ OT &= 10 \end{aligned}$$

Also $\sin TOX = 6/10 = 0.6$; $\cos TOX = 8/10 = 0.8$; $\tan TOX = 6/8 = 0.75$.

138. Vector Addition and Subtraction.—The resultant of any number of vectors may be found by taking the algebraic

sum of the horizontal and vertical components of the individual vectors. The horizontal and vertical components must be added separately, and the calculated sums represent the horizontal and vertical components of the resultant vector. The difference between two vectors may be obtained by subtracting, according to the rules of algebra, the components of one vector from those of another.

Example 1. Find the resultant, or vector sum, of the vectors: $\overline{OA} = -9 - j14$, $\overline{OB} = 15 + j12$, $\overline{OC} = 25 - j16$, $\overline{OD} = -12 + j6$. Calculate the numerical or scalar value of the resultant.

Solution:

$$\begin{aligned}
\overline{OA} &= -9 - j14 \\
\overline{OB} &= 15 + j12 \\
\overline{OC} &= 25 - j16 \\
\overline{OD} &= -12 + j\,6 \\
\hline
\bar{X} &= 19 - j12 \\
X^2 &= (19)^2 + (12)^2 \\
X^2 &= 505 \\
X &= 22.5 \textit{ Ans.}
\end{aligned}$$

Example 2. Subtract $\overline{OA} = 75 - j150$ from $\overline{OB} = -132 + j60$ and calculate the numerical or scalar value of the vector difference.

Solution:

$$\begin{aligned}
\overline{OB} &= -132 + j\,60 \\
\overline{OA} &= 75 - j150 \\
\hline
\bar{X} &= -207 + j210 \\
X^2 &= (207)^2 + (210)^2 \\
X^2 &= 86{,}949 \\
X &= 294.9 \textit{ Ans.}
\end{aligned}$$

Example 3. Add the vectors $\overline{OA} = -12 + j100$ and $\overline{OB} = 140 - j83$. From this sum subtract the vector $\overline{OC} = 128 + j227$.

Solution:

$$\begin{aligned}
\overline{OA} &= -12 + j100 \\
\overline{OB} &= 140 - j\,83 \\
\hline
\overline{OR} &= 128 + j\,17 \\
\overline{OC} &= 128 + j227 \\
\hline
\bar{X} &= 0 - j210 \\
X &= 210 \textit{ Ans.}
\end{aligned}$$

Problems

Plot the vectors represented by each of the following complex expressions and calculate the numerical or scalar value of each vector.

1. $3 + j4$	5. $-16 + j12$	9. $-11 - j60$
2. $12 - j5$	6. $20 + j21$	10. $-27 + j36$
3. $-24 + j7$	7. $-15 - j8$	11. $14 + j48$
4. $-9 - j40$	8. $-15 + j36$	12. $16 - j30$

13. Add vectors 1 and 2 and calculate the scalar value of the resultant vector.

14. Add vectors 3, 4, and 5 and calculate the scalar value of the resultant vector.

15. Subtract vector 6 from vector 5 and calculate the scalar value of this vector difference.

16. Subtract vector 9 from vector 8 and calculate the scalar value of this vector difference.

17. Add vectors 10 and 11, then subtract vector 12. What is the scalar value of the resultant vector?

18. From the sum of vectors 7 and 8 subtract the sum of vectors 11 and 12. Calculate the scalar value of the resultant.

19. Add vectors 5, 6 and 7, then subtract vector 4. Calculate the scalar value of the resultant vector.

20. Find the sum of vectors 9, 10, 11, and 12. Calculate the scalar value of this vector sum.

139. Polar Method of Representation.—It is often convenient to describe a vector by stating its length and the angle between it and some other line. From such a statement the complex expression for the vector can be readily obtained.

Example 4. A vector, 60 units long, is located so that the angle between it and the reference line OX is 150°. What is the complex expression of this vector?

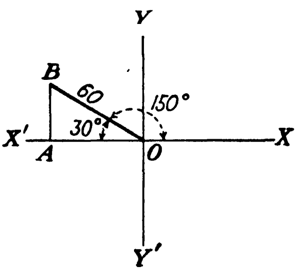


FIG. 69.

Solution:
In Fig. 69,

$$OA = -60 \cos 30°$$

and

$$AB = 60 \sin 30°$$

Therefore, the complex expression for vector OB is

$$\overline{60} = -60 \cos 30° + j60 \sin 30°$$
$$\overline{60} = -60\,(0.866) + j60(0.5)$$
$$\overline{60} = -52 + j30 \; Ans.$$

Problems

Plot each of the following vectors and write the complex expression for each:

1. Vector OA, 100 units long, making an angle of 25° with OX.
2. Vector OB, 50 units long, making an angle of 40° with OX.
3. Vector OC, 25 units long, making an angle of 125° with OX.
4. Vector OD, 150 units long, making an angle of 150° with OX.
5. Vector OE, 200 units long, making an angle of 208° with OX.
6. Vector OF, 250 units long, making an angle of 227° with OX.
7. Vector OG, 40 units long, making an angle of 260° with OX.
8. Vector OH, 70 units long, making an angle of 288° with OX.
9. Vector OM, 110 units long, making an angle of 305° with OX.
10. Vector ON, 60 units long, making an angle of 340° with OX.

140. Impedance as a Complex Quantity.—Electrical quantities can very conveniently be represented by means of complex quantities. We may consider the impedance of a circuit as a vector quantity and express it in the complex form. The conventional forms for the complex impedance expressions are:

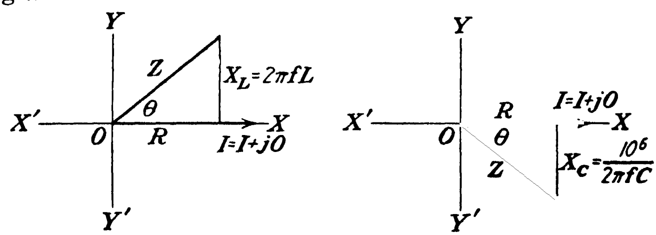
$$\bar{Z} = R + jX$$

where X is the inductive reactance, and

$$\bar{Z} = R - jX$$

where X is the capacitive reactance.

For these expressions the current is generally assumed to be taken along the reference line, as indicated in the following diagrams:



Impedance triangle for an inductive circuit

Impedance triangle for a capacitive circuit

• FIG. 70.

The complex expression for the total impedance of a series circuit is found by adding the complex expressions for the units which are connected in series. The numerical, or scalar,

values of impedance cannot be added directly in order to obtain the total impedance, unless they all have the same power factor. This, of course, is usually not the case. The following illustrative examples should serve to make this clear.

Example 5. A coil whose resistance is 8 ohms and whose inductance is 0.1 henry is connected in series with a noninductive resistance of 20 ohms and a second coil whose resistance is 12 ohms and whose inductance is 0.03 henry. Find the total impedance of this circuit at 25 cycles.

Solution: 1. Find the reactance of each coil at 25 cycles.

$$\begin{aligned} X_L &= 2\pi f L \\ X_1 &= 157(0.1) \\ X_1 &= 15.7 \\ X_2 &= 157(0.03) \\ X_2 &= 4.71 \end{aligned}$$

2. Write the complex expression for the impedance of each unit; then add them.

$$\begin{aligned} \text{For coil 1, } \bar{Z}_1 &= 8 + j15.7 \\ \text{For coil 2, } \bar{Z}_2 &= 12 + j\ 4.71 \\ \text{For the resistance, } \bar{Z} &= \underline{20 + j\ 0} \\ \text{The total impedance, } \bar{Z}_t &= 40 + j20.41 \\ \bar{Z}_t^2 &= (40^2) + (20.4)^2 \\ \bar{Z}_t^2 &= 2{,}016.16 \\ \bar{Z}_t &= 44.9 \text{ ohms } \textit{Ans.} \end{aligned}$$

Example 6. A condenser whose capacitance is 60 mf. is connected in series with a 40-ohm noninductive resistance and an impedance coil of 0.08-henry inductance and 10-ohm resistance. What will be the current if the voltage impressed on this circuit is 220 volts, 60 cycles?

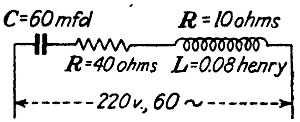


FIG. 71.

Solution: 1. Find the reactance values for the coil and the condenser.

$$\begin{aligned} X_L &= 2\pi f L \\ X_L &= 377(0.08) \\ X_L &= 30.16 \end{aligned}$$

or

$$\begin{aligned} X_L &= 30.2 \\ X_c &= \frac{10^6}{2\pi f C} \\ X_c &= \frac{1{,}000{,}000}{377(60)} \\ X_c &= 44.2 \end{aligned}$$

2. Find the total impedance of the circuit.

$$
\begin{aligned}
\text{For the condenser, } \bar{Z} &= 0 - j44.2 \\
\text{For the coil, } \bar{Z} &= 10 + j30.2 \\
\text{For the resistance, } \bar{Z} &= 40 + j\,0 \\
\text{For the circuit, } \bar{Z}_t &= 50 - j14 \\
Z_t^2 &= 2{,}500 + 196 \\
Z_t &= 51.9 \text{ ohms}
\end{aligned}
$$

3. Find the current

$$
\begin{aligned}
E &= IZ \\
220 &= I(51.9) \\
I &= \frac{220}{51.9} = 4.24 \text{ amp. } \textit{Ans.}
\end{aligned}
$$

Example 7. What is the power factor of the circuit described in Ex. 6? How much power is used and what is the e.m.f. across each unit?

Solution:

$$
\begin{aligned}
\text{Power factor } &= \cos\theta = \frac{R}{Z} = \frac{50}{51.9} = 0.963 \textit{ Ans.} \\
\text{Power used } &= W = EI\cos\theta \\
W &= 220(4.24)(0.963) \\
W &= 8{,}980 \text{ watts } \textit{Ans.}
\end{aligned}
$$

The e.m.f. across each unit is found by using $E = IZ$.
The impedance of the condenser is 44.2 ohms.

$$
\begin{aligned}
E &= IZ \\
E &= 4.24(44.2) \\
E &= 187.4 \text{ volts across the condenser } \textit{Ans.}
\end{aligned}
$$

The impedance of the noninductive resistance is 40 ohms.
$\therefore E = 4.24(40) = 169.6$ or 170 volts across the resistance *Ans.*
The impedance of the coil must be calculated

$$
\begin{aligned}
Z^2 &= R^2 + X^2 \\
Z^2 &= (10)^2 + (30.2)^2 \\
Z^2 &= 100 + 912.04 \\
Z^2 &= 1{,}012.04 \\
Z &= 31.81
\end{aligned}
$$

$$
\begin{aligned}
E &= IZ \\
E &= 4.24(31.81) \\
E &= 134.9 \text{ volts across the coil } \textit{Ans.}
\end{aligned}
$$

The circuit e.m.f. of 220 volts is the *vector* sum of the three voltages 187.4, 17, and 134.9.

Problems

1. A coil whose resistance is 10 ohms and whose inductance is 0.01 henry is connected in series with a 100-mf. condenser. Find the impedance of the circuit and the current when the impressed e.m.f. is 110 volts, 60 cycles.

2. Find the voltage across each part of the circuit in Prob. 1.

3. A condenser whose capacitance is 50 mf. is connected in series with a 20-ohm noninductive resistance and an impedance coil of 0.15-henry inductance and 2-ohm resistance. What is the impedance of this circuit at 60 cycles? What current will flow when the impressed e.m.f. is 120 volts, 60 cycles?

4. Find the e.m.f. across each part of the circuit described in Prob. 3.

5. A coil of 0.25-henry inductance and 15-ohm resistance is connected in series with a 150-mf. condenser. Find the current through this series combination if the impressed e.m.f. is (*a*) 110 volts, 25 cycles and (*b*) 110 volts, 60 cycles.

6. A coil whose inductance is 0.035 henry and whose resistance is 38 ohms, is in series with a 20-mf. condenser and a 17-ohm noninductive resistance. Find the current (*a*) at 110 volts, 60 cycles and (*b*) at 110 volts, 150 cycles.

7. A coil whose resistance is 4 ohms and whose inductance is 0.0002 henry, is connected in series with a 0.0015-mf. condenser. What is the impedance of this combination (*a*) at 100 kilocycles and (*b*) at 500 kilocycles?

8. An inductance of 0.005 henry and negligible resistance is connected in series with two condensers whose capacitances are 2 and 3 mf., respectively. What is the impedance of this circuit at 10,000 cycles? What are the current and the voltage across each unit when the impressed e.m.f. is 50 volts, 10,000 cycles?

Example 8. A coil of 5-ohm resistance and 0.02-henry inductance is connected in series with a 4-mf. condenser. What is the smallest condenser that can be connected in series with this circuit to reduce its impedance to 8 ohms at 1,000 cycles and keep the circuit inductive?

Solution: 1. Find the impedance of the original circuit.

$$X_L = 6{,}280(0.02)$$
$$= 125.6$$
$$X_c = \frac{10^6}{6{,}280(4)}$$
$$= 39.6$$

$$\begin{array}{lcl} Z \text{ (coil)} & = & 5 + j125.6 \\ \underline{Z \text{ (condenser)}} & = & \underline{0 - j\ 39.6} \\ Z & = & 5 + j\ 86 \end{array}$$

2 Determine the complex expression for the impedance of the required circuit

$$
\begin{aligned}
Z^2 &= R^2 + X^2 \\
(8)^2 &= (5)^2 + X^2 \\
64 &= 25 + X^2 \\
39 &= X^2 \\
X &= 6.25 \text{ ohms}
\end{aligned}
$$

The required circuit should have a reactance of 6.25 ohms and a resistance of 5 ohms The complex expression for its impedance is, therefore

$$\bar{8} = 5 + j6.25$$

3 Determine the reactance of the condenser needed by subtracting the impedance of the original circuit from that of the required circuit

$$
\begin{aligned}
\bar{8} &= 5 + j\,6.25 \\
\bar{Z} &= 5 + j86 \\
\hline
\bar{Z} &= 0 - j79.75
\end{aligned}
$$

which means that the reactance of the condenser needed is 79.75 ohms

4 Find the capacitance of the condenser

$$
\begin{aligned}
X_c &= \frac{10^6}{2\pi f C} \\
79.75 &= \frac{10^6}{6{,}280(C)} \\
79.75(6{,}280)C &= 10^6 \\
C &= \frac{1{,}000{,}000}{79.75(6{,}280)} \\
C &= 2 \text{ mf., approximately } \textit{Ans.}
\end{aligned}
$$

9. What size condenser should be connected in series with a coil of 10-ohm resistance and 0.05-henry inductance to reduce the impedance of the combination to 12.5 ohms at a frequency of 60 cycles, keeping the total reactance inductive?

10. What size condenser could be used in Prob. 9 if the total reactance may be capacitive?

11. A 0.5-mf. condenser is connected in series with a coil of negligible resistance. What is the inductance of the coil if the impedance of the combination at 50,000 cycles is 3.5 ohms?

12. A 0.0002-mf. condenser is in series with an inductance coil whose resistance is 1 ohm. What is the inductance of the coil if the total impedance of the circuit at 1,000 kilocycles is 2.6 ohms?

141. Leading and Lagging Currents.—When a circuit contains inductance or capacitance the current in the circuit is

not in phase with the voltage which produces it. By this is meant that at the instant when the voltage is zero the current which it produces is not zero, or when the voltage is at its maximum value, the current has a value which differs from its maximum current value.

When a given voltage reaches its maximum value some time before the maximum current value occurs, the current is said to *lag*, and when a given voltage reaches its maximum value some time after the maximum current value has been reached, the current is said to *lead*. Note that the words *leading* and *lagging* always refer to the position of the current with reference to the voltage. For example, when a voltage is described as an e.m.f. of 110 volts, power factor 60 per cent lagging, we mean that the current produced by this voltage lags behind it by an angle whose cosine is 0.60.

Example 9. The current in a circuit is 40 amp. If the e.m.f. of the circuit is expressed by $120 + j0$, what is the complex expression for this current when the power factor is (*a*) 92 per cent leading, and (*b*) 72 per cent lagging?

Solution: (*a*) From the Power-factor Table on page 249 we see that when $\cos\theta = 0.92$, $\sin\theta = 0.3919$. Therefore, the complex expression for the current in this case is

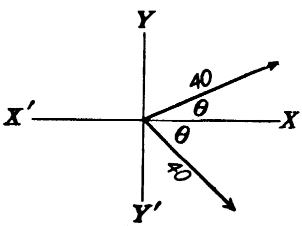


FIG. 72.

$$\overline{40} = 40(0.92) + j40(0.3919)$$
$$\overline{40} = 36.8 + j15.68 \ Ans.$$

(*b*) Similarly, when $\cos\theta = 0.72$, $\sin\theta = 0.694$

$$\therefore \overline{40} = 40(0.72) - j40(0.694)$$
$$\overline{40} = 28.8 - j27.8 \ Ans.$$

Problems

1. The current in a circuit is 50 amp. If the e.m.f. of the circuit is taken as $220 + j0$, what is the complex expression for the current when the power factor is (*a*) 95 per cent lagging, (*b*) 90 per cent lagging, (*c*) 80 per cent leading, (*d*) 70 per cent leading, (*e*) 65 per cent lagging, and (*f*) 50 per cent leading?

2. The e.m.f. of a circuit is 220 volts. If the current in the circuit is represented by $10 + j0$, what is the complex expression for this e.m.f. if the power factor of the circuit is (*a*) 85 per cent leading, (*b*) 75 per cent lagging, (*c*) 60 per cent lagging, (*d*) 55 per cent leading, and (*e*) 40 per cent lagging?

3. What is the complex expression for the current in a circuit in which the e.m.f. is given as $220 + j0$ if the resistance of the circuit is 10 ohms and its inductive reactance is 7.5 ohms?

4. The e.m.f. across a circuit whose resistance is 12 ohms and whose inductive reactance is 5 ohms is represented by $260 + j0$. What is the complex expression for the current in the circuit?

5. A circuit has a resistance of 40 ohms and a capacitive reactance of 9 ohms. If the e.m.f. across the circuit is represented by $1{,}230 + j0$, write the complex expression for the current.

6. What is the complex expression for an e.m.f. of 200 volts impressed on a circuit whose resistance is 12 ohms and whose inductive reactance is 15 ohms, the current in the circuit being represented by $I + j0$?

7. Repeat Prob. 6 for an e.m.f. of 250 volts impressed on a circuit having a resistance of 50 ohms and a capacitive reactance of 120 ohms.

8. A circuit has a total resistance of 12 ohms and an inductive reactance of 7 ohms. Write the complex expression for the e.m.f. across the circuit if the current is represented by $15 + j0$. What is the effective value of this e.m.f.?

9. The current in a certain circuit is 50 amp. Write the complex expression for the current, taking the voltage as $E + j0$ (*a*) when θ is 15° and the current is leading; (*b*) when θ is 30°, current lagging; (*c*) when θ is 60°, current lagging; and (*d*) when θ is 40°, current leading.

Example 10. Two coils are in series with their voltages 46° 18′ out of phase. The e.m.f. across the first is 110 volts; across the second it is 85 volts. Find the e.m.f. across the series combination.

Solution: 1. Draw the vector diagram, placing one of the voltages along the reference line.

2. Write the complex expression for each vector and add them. This sum will be the complex expression for the total voltage

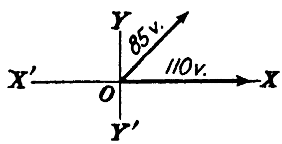


FIG. 73.

$$\overline{85} = 85 \cos 46° \, 18' + j85 \sin 46° \, 18'$$
$$\overline{85} = 85(0.69088) + j85(0.72297)$$

$$\begin{aligned}
\overline{85} &= 58.7 + j61.5 \\
\overline{110} &= 110 \quad + j\,0 \\
\hline
\bar{E} &= 168.7 + j61.5 \\
E^2 &= (168.7)^2 + (61.5)^2 \\
E^2 &= 28{,}460 + 3{,}782 \\
E^2 &= 32{,}242 \\
E &= 179.6 \text{ volts } \textit{Ans.}
\end{aligned}$$

Example 11. The total current supplied to two branches of a parallel circuit is 50 amp., power factor 80 per cent lagging. One of the branch currents has a value of 25 amp., power factor 86 per cent leading. Find the current in the second branch and its power factor.

Solution: 1. Draw the vector diagram and write the complex expression for each vector, assuming the voltage to be along the reference line.

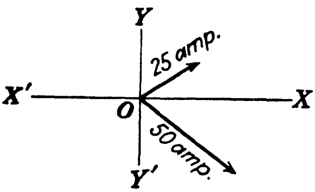


FIG. 74.

$$\overline{50} = 50(0.80) - j50(0.60)$$
$$\overline{25} = 25(0.86) + j25(0.51)$$

2. Since the sum of the two branch currents is 50 amp., we find the complex expression for the second branch current by subtracting the complex expression for the first branch current from that of the total current, as follows:

$$\overline{50} = 40 - j30 \quad \text{(expression for total current)}$$
$$\underline{\overline{25} = 21.5 + j12.75} \quad \text{(expression for current in branch 1)}$$
$$\bar{I} = 18.5 - j42.75 \quad \text{(expression for current in branch 2)}$$
$$I^2 = 342.25 + 1{,}827.56$$
$$I^2 = 2{,}169.81$$
$$I = 46.6 \text{ amp.}$$

$$\cos\theta, \text{ or power factor } = \frac{18.5}{46.6} = 39.7 \text{ per cent, lagging } \textit{Ans.}$$

10. Two coils, connected in series, have voltages which are 25° 11′ out of phase. One of the coils has an e.m.f. of 75 volts, the other of 47 volts. Write the complex expression for each of these voltages, taking one of them along the reference line. Find the complex expression for the total e.m.f. and its numerical or scalar value.

11. The currents in two parallel branches of a circuit are 14.7 amp. and 13.6 amp., and the phase difference between the two currents is 36° 8′. Find the current in the line feeding the two branches.

12. A generator supplies current to two motors connected in parallel. The first motor takes a current of 42.7 amp. at 90 per cent power factor. The second motor takes 31.2 amp. at 60 per cent power factor. Find the generator current and its power factor. Both motors take lagging currents.

13. Two coils connected in series have voltages of 213 volts and 625 volts which differ in phase by 21° 12′. Find the total e.m.f. across the two coils.

14. The current in a circuit is 32.7 at 95 per cent power factor, lagging. The current in a second circuit is 45 amp. at 40 per cent power factor, lagging. What is the current in the line which supplies current to these two circuits and what is its power factor?

15. The total current supplied to two coils in parallel is 36 amp. at 70 per cent power factor, lagging. One of the coils takes 20 amp. at 90 per cent power factor, lagging. What are the current and power factor of the second coil?

16. The total e.m.f. across two coils in series is 285 volts, power factor 85 per cent lagging. One of the two coils has an e.m.f. of 112 volts, power factor 70 per cent lagging. What are the voltage and power factor of the second coil?

17. An inductive branch of a circuit takes 15 amp. at 90 per cent power factor. In parallel with it is a capacitive branch which takes 25 amp. at 90 per cent power factor. What is the total current and what is its power factor? Does the total current lead or lag with reference to the line voltage?

18. A coil is in parallel with a circuit consisting of a resistance in series with a condenser. The coil draws 18 amp. at 70 per cent power factor. The condensive circuit draws 12 amp. at 50 per cent power factor. What is the total line current and what is its power factor?

19. In a circuit as described in Prob. 18 the total line current is 55 amp., power factor 80 per cent lagging. If the current through the coil is 40 amp. at a 40 per cent lagging power factor, what is the current through the second branch and what is its power factor?

20. Repeat Prob. 19 with the total current 55 amp. at 90 per cent power factor, leading, and all other values unchanged.

21. The currents in two parallel branches of a circuit are 51.7 amp. in the first and 39.5 amp. in the second. If the phase difference between these two currents is 90°, what is the current in the line supplying the two branches?

22. Repeat Prob. 21 for currents whose phase difference is 120°.

23. Two coils whose voltages are 90° out of phase are connected in series. Find the total e.m.f. across the two coils if the first has an e.m.f. of 235 volts and the second 147 volts.

24. Repeat Prob. 23 for two coils whose voltages are out of phase by 120°.

25. The currents in three parallel branches of a circuit are

15 amp. in branch 1
25 amp. in branch 2, lagging 15° 12′ behind 1
30 amp. in branch 3, lagging 29° 46′ behind 1

Find the total current.

26. The currents in three parallel branches of a circuit are

21 amp. in branch 1, power factor 100 per cent
20 amp. in branch 2, power factor 96 per cent lagging
32 amp. in branch 3, power factor 87 per cent leading

Find the total current.

27. Solve Prob. 26 using the following values:

17 amp. in branch 1
38 amp. in branch 2, lagging 35° 12′ behind 1
46 amp. in branch 3, leading 1 by 78° 11′

28. Solve Prob. 26 using the following values:

18 amp. in branch 1, power factor 100 per cent
45 amp. in branch 2, power factor 78 per cent lagging
40 amp. in branch 3, power factor 70 per cent lagging

29. Repeat Prob. 26 using the following values:

40 amp. in branch 1
50 amp. in branch 2, lagging 38° 45′ behind 1
65 amp. in branch 3, leading 1 by 42° 38′

30. Repeat Prob. 26 using the following values:

50 amp. in branch 1, power factor 100 per cent
20 amp. in branch 2, power factor 70 per cent lagging
100 amp. in branch 3, power factor 90 per cent leading

31. Repeat Prob. 26 using the following values:

60 amp. in branch 1, leading 2 by 67° 28′
75 amp. in branch 2, lagging 45° 38′ behind 3
85 amp. in branch 3

32. Repeat Prob. 26 using the following values:

75 amp. in branch 1, leading 3 by 80° 27′
80 amp. in branch 2, lagging 135° 12′ behind 1
100 amp. in branch 3

142. Complex Expression for Power.—The apparent power in an alternating circuit may also be represented by a complex expression. The general expression for a current which leads is

$$\bar{I} = I \cos \theta + jI \sin \theta.$$

In order to obtain a complex expression for the power in the circuit which carries this current, we may multiply the expression by E, the e.m.f. of the circuit. This gives

$$\overline{EI} = EI \cos \theta + jEI \sin \theta$$

as an expression for the apparent power of a circuit in which the current leads the voltage. The corresponding expression for the apparent power of a circuit, in which the current lags, is

$$\overline{EI} = EI \cos \theta - jEI \sin \theta$$

When a group of lamps is connected to a circuit it is usual to consider that the current through the lamps is in phase with the voltage across them. In other words, the power factor of the lamp group is 100 per cent, or unity.

When the statement is made that a certain motor draws 6 kva. from a power line we know that this is the apparent power taken by this motor, since kilovolt-amperes always refers to the apparent power. On the other hand, when we say a certain motor draws 6 kw., we mean that the true power taken by the motor is 6 kw. In the first case $EI = 6$ (E expressed in kilovolts), in the second case $EI \cos\theta = 6$. This is a very important distinction.

Example 12. A motor uses 37 kw. at 87 per cent power factor. A group of lamps on the same circuit draws 50 amp. The e.m.f. of the circuit is 220 volts. Find the total effective power, reactive power, and power factor.

Solution. 1. Determine the complex expression for the power taken by the motor. Since

$$EI\,(0.87) = 37$$
$$EI = 37 \div 0.87 = 42.5$$

Hence the desired expression is

$$\overline{42.5} = 37 - j42.5(0.493)$$

or

$$\overline{42.5} = 37 - j21$$

2. Add the complex expression for the power taken by the lamps. The power for the lamps is

$$220(50) = 11{,}000 \text{ watts or } 11 \text{ kw.}$$

$$\begin{aligned} \overline{EI} &= 11 + j\,0 \\ \overline{42.5} &= 37 - j21 \\ \hline \overline{EI} &= 48 - j21 \\ EI &= 52.4 \end{aligned}$$

The complex expression $\overline{EI} = 48 - j21$ tells us that the total effective power is 48 kw. and that the total reactive power is 21 kva.

3. The power factor for the whole circuit is

$$\cos\theta = \frac{48}{52.4} = 91.6 \text{ per cent } \textit{Ans.}$$

Example 13. Two motors on the same line together use 33 kw. at 96 per cent power factor, leading. One of the motors draws 25 kw. at

86 per cent power factor, lagging. What is the apparent power taken by the second motor and what is its power factor?

Solution: Write the complex expressions for the total apparent power and that of the first motor using

$$EI = EI \cos \theta \pm jEI \sin \theta$$

For the whole circuit

$$EI(0.96) = 33$$
$$EI = 34.4$$

For motor 1

$$EI(0.86) = 25$$
$$EI = 29.1$$

$$\overline{34.4} = 33 + j34.4(0.28)$$
$$\overline{29.1} = 25 - j29.1(0.51)$$

Next, subtract the power taken by the motor from that of the whole circuit.

$$\begin{aligned} \overline{34.4} &= 33 + j\ 9.63 \\ \overline{29.1} &= 25 - j14.84 \\ \hline \overline{EI} &= 8 + j24.47 \\ (EI)^2 &= 64 + 600.25 \\ (EI)^2 &= 664.25 \\ EI &= 25.8 \text{ kva. } Ans. \end{aligned}$$

$$\text{Power factor, or } \cos \theta = \frac{8}{25.8} = 31 \text{ per cent, leading } Ans.$$

Problems

1. Find the current in a circuit in which 38.5 kw. are being used at 85 per cent power factor and 220 volts.

2. A motor delivers 6 hp. and is 85 per cent efficient at that load. On the same circuit there is a group of lamps using 4.5 kw. The total power factor is 92 per cent. Find the e.m.f. of the circuit if the current is 45 amp.

3. Find the power factor of a circuit in which a motor is drawing 16 amp. at 225 volts and 82 per cent power factor. A group of lamps on the same circuit is drawing 20 amp. Also find the total current.

4. A motor is drawing 25 amp. from a 220-volt line at 75 per cent power factor and a group of lamps on the same circuit uses 23 amp. Find the total effective power, power factor, and total current.

5. A motor takes 18 kw. at 80 per cent power factor. A group of lamps on the same circuit draws 5 kw. Find the total effective power, apparent power and total power factor.

6. A group of lamps is drawing 16 amp. from a 110-volt line. A motor on the same circuit is drawing 45 amp. at 70 per cent power factor. Find the total current, power factor, and total apparent power.

7. The current in a line supplying a motor and a group of lamps, connected in parallel, is 103 amp. The lamps draw 45 amp. How much

current is the motor drawing and at what power factor? The power factor of the line current is 91 per cent.

8. A motor delivers 8 hp., and its efficiency is 85 per cent. A group of lamps on the same line uses 3.6 kw. The total line current is 60 amp., and its power factor 90 per cent. Find the e.m.f. of the circuit.

9. A 20-hp. motor, efficiency 90 per cent, power factor 70 per cent, is operating on a 2,200-volt line. Find the line current.

10. A motor draws 38 amp. at 85 per cent power factor. A group of lamps on the same circuit causes the total line current to be raised to 57.8 amp. at 94 per cent power factor. How much current do the lamps take from the line?

11. A motor is connected in parallel with a group of lamps. The total line current is 75 amp., and its power factor 80 per cent. If the lamps draw 7 kw. and the motor delivers 12 hp. and is 90 per cent efficient, what is the e.m.f. of the line?

12. The current in a line supplying a motor and a group of lamps in parallel is 76 amp. at 90 per cent power factor. The lamps draw 35 amp. How much current does the motor draw and at what power factor?

13. A motor draws 46 amp. at 88 per cent power factor. A group of lamps on the same circuit causes the total line current to be 70 amp. at 95 per cent power factor. How much current do the lamps draw?

14. Two motors are connected in parallel across the same line. The first draws 25 kw. at 75 per cent power factor, and the second 25 kva. at 85 per cent power factor. Both currents lag, and the line e.m.f. is 1,100 volts. Find the total effective power and the line current.

15. A motor draws 21 kw. at 80 per cent lagging power factor from a 2,200-volt line. Another motor on the same line draws 28 kva. at 85 per cent leading power factor. Find the total apparent power, power factor, and line current.

16. The current in a line supplying a motor and a group of lamps in parallel is 86 amp., power factor 84 per cent. The motor draws 71.4 amp. at 75.7 per cent power factor. How much current do the lamps draw?

17. A generator supplies power to two motors connected in parallel. One motor uses 32 kva. at 90 per cent power factor, and the other 25 kw. at 86 per cent. The generator e.m.f. is 1,100 volts, and both motors take lagging currents. If the generator is operating at full load, what is its rating in kilovolt-amperes?

18. A group of motors taking 30 kva. at 76 per cent lagging power factor is connected to the same circuit with a group of lamps drawing 70 amp. The e.m.f. of the circuit is 110 volts. A miscellaneous load of 10 kw. at 90 per cent lagging power factor is also connected to this circuit. What must be the rating in kilovolt-amperes of the generator which is to supply this load?

19. An a.-c. generator supplies the following load: 100 kva. at 80 per cent power factor, 130 kw. at 65 per cent power factor, 75 kw. at 100 per

cent power factor, and 150 kw. at 90 per cent power factor. If all currents are lagging, what must be the rating of the generator in kilovolt-amperes? What is the total kilowatt load, and what is the power factor?

20. A synchronous motor which draws 27 kw. at a leading power factor of 80 per cent is connected in parallel with an induction motor taking 30 kva. at 70 per cent power factor, lagging. If the line e.m.f. is 2,200 volts, what are the total apparent power, power factor, and line current?

21. A 10-hp. motor, a 15-hp. motor, and a 7.5-hp. motor are operating from the same line. The first is running at one-half load, 75 per cent efficiency and at 70 per cent lagging power factor; the second is operating at full load, 90 per cent efficiency, and 85 per cent lagging power factor; and the third at 80 per cent load, 85 per cent efficiency and 82 per cent leading power factor. What is the rating in kilovolt-amperes and kilowatts of the generator which supplies current to these motors, if this load represents its rated output? What is the power factor of the generator with this load?

APPENDIX

APPENDIX

Complete Wire Table, Standard Annealed Copper American Wire Gage (B & S)

Gage number	Diameter, mils at 20° C	Circular mils	Ohms per 1,000 ft., 20° C (68° F)
0000	460 0	211,600	0 04901
000	409 6	167,800	0 06180
00	364 8	133,100	0 07793
0	324 9	105,500	0 09827
1	289 3	83,690	0 1239
2	256 6	66,370	0 1563
3	229 4	52,640	0 1970
4	204 3	41,740	0 2485
5	181 9	33,100	0 3133
6	162 0	26 250	0 3951
7	144 3	20,820	0 4982
8	128 5	16,510	0 6282
9	114 4	13,090	0 7921
10	101 9	10,380	0 9989
11	90 74	8,234	1 260
12	80 81	6 530	1 588
13	71 96	5,178	2 003
14	64 08	4,107	2 525
15	57 07	3,257	3 184
16	50 82	2,583	4 016
17	45 26	2,048	5 064
18	40 30	1,624	6 385
19	35 89	1,288	8 051
20	31 96	1,022	10 15
21	28 46	810 1	12 80
22	25 35	642 4	16 14
23	22 57	509 5	20 36
24	20 10	404 0	25 67
25	17 90	320 4	32 37
26	15 94	254 1	40 81
27	14 20	201 5	51 47
28	12 64	159 8	64 90
29	11 26	126 7	81 83
30	10 03	100 5	103 2
31	8 928	79 70	130 1
32	7 950	63 21	164 1
33	7 080	50 13	206 9
34	6 305	39 75	260 9
35	5 615	31 52	329 0
36	5 000	25 00	414 8
37	4 453	19 83	523 1
38	3 965	15 72	659 6
39	3 531	12 47	831 8
40	3.145	9 888	1,049 0

′	N Sin	N Tan	N Cot	N Cos	
0	00000	00000	∞	1 0000	**60**
1	029	029	3437 7	000	59
2	058	058	1718 9	000	58
3	087	087	1145 9	000	57
4	116	116	859 44	000	56
5	00145	00145	687 5$\bar{5}$	1 0000	**55**
6	17$\bar{5}$	17$\bar{5}$	572 96	000	54
7	204	204	491 11	000	53
8	233	233	429 72	000	52
9	262	262	381 97	000	51
10	00291	00291	343 77	1 0000	**50**
11	320	320	312 52	99999	49
12	349	349	286 48	999	48
13	378	378	264 44	999	47
14	407	407	245 55	999	46
15	00436	00436	229 18	99999	**45**
16	465	465	214 86	999	44
17	49$\bar{5}$	49$\bar{5}$	202 22	999	43
18	524	524	190 98	999	42
19	553	553	180 93	998	41
20	00582	00582	171 89	99998	**40**
21	611	611	163 70	998	39
22	640	640	156 26	998	38
23	669	669	149 47	998	37
24	698	698	143 24	998	36
25	00727	00727	137 51	99997	**35**
26	756	756	132 22	997	34
27	785	785	127 32	997	33
28	814	81$\bar{5}$	122 77	997	32
29	844	844	118 54	996	31
30	00873	00873	114 59	99996	**30**
31	902	902	110 89	996	29
32	931	931	107 43	996	28
33	960	960	104 17	995	27
34	00989	00989	101 11	995	26
35	01018	01018	98 218	9999$\bar{5}$	**25**
36	047	047	95 489	99$\bar{5}$	24
37	076	076	92 908	994	23
38	105	105	90 463	994	22
39	134	13$\bar{5}$	88 144	994	21
40	01164	01164	85 940	99993	**20**
41	193	193	83 844	993	19
42	222	222	81 847	993	18
43	251	251	79 943	992	17
44	280	280	78 126	992	16
45	01309	01309	76 390	99991	**15**
46	338	338	74 729	991	14
47	367	367	73 139	991	13
48	396	396	71 615	990	12
49	425	425	70 153	990	11
50	01454	0145$\bar{5}$	68 750	99989	**10**
51	483	484	67 402	989	9
52	513	513	66 105	989	8
53	542	542	64 858	988	7
54	571	571	63 657	988	6
55	01600	01600	62 499	99987	**5**
56	629	629	61 383	987	4
57	658	658	60 306	986	3
58	687	687	59 266	986	2
59	716	716	58 261	985	1
60	01745	01746	57 290	9998$\bar{5}$	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	01745	01746	57 290	9998$\bar{5}$	**60**
1	774	77$\bar{5}$	56 351	984	59
2	803	804	55 442	984	58
3	832	833	54 561	983	57
4	862	862	53 709	983	56
5	01891	01891	52 882	99982	**55**
6	920	920	52 081	982	54
7	949	949	51 303	981	53
8	01978	01978	50 549	980	52
9	02007	02007	49 816	980	51
10	02036	02036	49 104	99979	**50**
11	065	066	48 412	979	49
12	094	09$\bar{5}$	47 740	978	48
13	123	124	47 085	977	47
14	152	153	46 449	977	46
15	02181	02182	45 829	99976	**45**
16	211	211	45 226	976	44
17	240	240	44 639	97$\bar{5}$	43
18	269	269	44 066	974	42
19	298	298	43 508	974	41
20	02327	02328	42 964	99973	**40**
21	356	357	42 433	972	39
22	385	386	41 916	972	38
23	414	41$\bar{5}$	41 411	971	37
24	443	444	40 917	970	36
25	02472	02473	40 436	99969	**35**
26	501	502	39 965	969	34
27	530	531	39 506	968	33
28	560	560	39 057	967	32
29	589	589	38 618	966	31
30	02618	02619	38 188	99966	**30**
31	647	648	37 769	96$\bar{5}$	29
32	676	677	37 358	964	28
33	70$\bar{5}$	706	36 956	963	27
34	734	735	36 563	963	26
35	02763	02764	36 178	99962	**25**
36	792	793	35 801	961	24
37	821	822	35 431	960	23
38	850	851	35 070	959	22
39	879	881	34 715	959	21
40	02908	02910	34 368	99958	**20**
41	938	939	34 027	957	19
42	967	968	33 694	956	18
43	02996	02997	33 366	955	17
44	0302$\bar{5}$	03026	33 045	954	16
45	03054	03055	32 730	99953	**15**
46	083	084	32 421	952	14
47	112	114	32 118	952	13
48	141	143	31 821	951	12
49	170	172	31 528	9$\bar{5}$0	11
50	03199	03201	31 242	99949	**10**
51	228	230	30 960	948	9
52	257	259	30 683	947	8
53	286	288	30 412	946	7
54	316	317	30 14$\bar{5}$	945	6
55	0334$\bar{5}$	03346	29 882	99944	**5**
56	374	376	29 624	943	4
57	403	40$\bar{5}$	29 371	942	3
58	432	434	29 122	941	2
59	461	463	28 877	940	1
60	03490	03492	28 636	99939	**0**
	N Cos	N Cot	N Tan	N Sin	′

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′	N Sin	N Tan	N Cot	N Cos	
0	.03490	.03492	28.636	.99939	**60**
1	519	521	.399	938	59
2	548	550	28.166	937	58
3	577	579	27.937	936	57
4	606	609	.712	935	56
5	.03635	.03638	27.490	.99934	**55**
6	664	667	.271	933	54
7	693	696	27.057	932	53
8	723	725	26.845	931	52
9	752	754	.637	930	51
10	.03781	.03783	26.432	.99929	**50**
11	810	812	.230	927	49
12	839	842	26.031	926	48
13	868	871	25.835	925	47
14	897	900	.642	924	46
15	.03926	.03929	25.452	.99923	**45**
16	955	958	.264	922	44
17	.03984	.03987	25.080	921	43
18	.04013	.04016	24.898	919	42
19	042	046	.719	918	41
20	.04071	.04075	24.542	.99917	**40**
21	100	104	.368	916	39
22	129	133	.196	915	38
23	159	162	24.026	913	37
24	188	191	23.859	912	36
25	.04217	.04220	23.695	.99911	**35**
26	246	250	.532	910	34
27	275	279	.372	909	33
28	304	308	.214	907	32
29	333	337	23.058	906	31
30	.04362	.04366	22.904	.99905	**30**
31	391	395	.752	904	29
32	420	424	.602	902	28
33	449	454	.454	901	27
34	478	483	.308	900	26
35	.04507	.04512	22.164	.99898	**25**
36	536	541	22.022	897	24
37	565	570	21.881	896	23
38	594	599	.743	894	22
39	623	628	.606	893	21
40	.04653	.04658	21.470	.99892	**20**
41	682	687	.337	890	19
42	711	716	.205	889	18
43	740	745	21.075	888	17
44	769	774	20.946	886	16
45	.04798	.04803	20.819	.99885	**15**
46	827	833	.693	883	14
47	856	862	.569	882	13
48	885	891	.446	881	12
49	914	920	.325	879	11
50	.04943	.04949	20.206	.99878	**10**
51	.04972	.04978	20.087	876	9
52	.05001	.05007	19.970	875	8
53	030	037	.855	873	7
54	059	066	.740	872	6
55	.05088	.05095	19.627	.99870	**5**
56	117	124	.516	869	4
57	146	153	.405	867	3
58	175	182	.296	866	2
59	205	212	.188	864	1
60	05234	.05241	19.081	.99863	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.05234	.05241	19.081	.99863	**60**
1	263	270	18.976	861	59
2	292	299	.871	860	58
3	321	328	.768	858	57
4	350	357	.666	857	56
5	.05379	.05387	18.564	.99855	**55**
6	408	416	.464	854	54
7	437	445	.366	852	53
8	466	474	.268	851	52
9	495	503	.171	849	51
10	.05524	.05533	18.075	.99847	**50**
11	553	562	17.980	846	49
12	582	591	.886	844	48
13	611	620	.793	842	47
14	640	649	.702	841	46
15	.05669	.05678	17.611	.99839	**45**
16	698	708	.521	838	44
17	727	737	.431	836	43
18	756	766	.343	834	42
19	785	795	.256	833	41
20	.05814	.05824	17.169	.99831	**40**
21	844	854	17 084	829	39
22	873	883	16.999	827	38
23	902	912	.915	826	37
24	931	941	.832	824	36
25	.05960	.05970	16.750	.99822	**35**
26	.05989	.05999	.668	821	34
27	.06018	.06029	.587	819	33
28	047	058	.507	817	32
29	076	087	.428	815	31
30	.06105	06116	16.350	.99813	**30**
31	134	145	.272	812	29
32	163	175	.195	810	28
33	192	204	.119	808	27
34	221	233	16.043	806	26
35	.06250	06262	15.969	.99804	**25**
36	279	291	.895	803	24
37	308	321	.821	801	23
38	337	350	.748	799	22
39	366	379	.676	797	21
40	.06395	.06408	15.605	.99795	**20**
41	424	438	.534	793	19
42	453	467	.464	792	18
43	482	496	.394	790	17
44	511	525	.325	788	16
45	.06540	.06554	15.257	.99786	**15**
46	569	584	.189	784	14
47	598	613	.122	782	13
48	627	642	15.056	780	12
49	656	671	14.990	778	11
50	.06685	.06700	14.924	.99776	**10**
51	714	730	.860	774	9
52	743	759	.795	772	8
53	773	788	.732	770	7
54	802	817	.669	768	6
55	.06831	.06847	14.606	.99766	**5**
56	860	876	.544	764	4
57	889	905	.482	762	3
58	918	934	.421	760	2
59	947	963	.361	758	1
60	.06976	.06993	14.301	.99756	**0**
	N Cos	N Cot	N Tan	N Sin	′

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′	N Sin	N Tan	N Cot	N Cos	
0	.06976	.06993	14.301	.99756	**60**
1	.07005̄	.07022	.241	754	59
2	034	051	.182	752	58
3	063	080	.124	750	57
4	092	110	.065	748	56
5	.07121	.07139	14.008	.99746	**55**
6	15̄0	168	13.951	744	54
7	179	197	.894	742	53
8	208	227	.838	740	52
9	237	256	.782	738	51
10	.07266	.07285	13.727	.99736	**50**
11	295̄	314	.672	734	49
12	324	344	.617	731	48
13	353	373	.563	729	47
14	382	402	.510	727	46
15	.07411	.07431	13.457	.99725	**45**
16	440	461	.404	723	44
17	469	490	.352	721	43
18	498	519	.300	719	42
19	527	548	.248	716	41
20	.07556	.07578	13.197	.99714	**40**
21	585̄	607	.146	712	39
22	614	636	.096	710	38
23	643	665	13.046	708	37
24	672	695̄	12.996	705	36
25	.07701	.07724	12.947	.99703	**35**
26	730	753	.898	701	34
27	759	782	.85̄0	699	33
28	788	812	.801	696	32
29	817	841	.754	694	31
30	.07846	.07870	12.706	.99692	**30**
31	875̄	899	.659	689	29
32	904	929	.612	687	28
33	933	958	.566	685̄	27
34	962	.07987	.520	683	26
35	.07991	.08017	12.474	.99680	**25**
36	.08020	046	.429	678	24
37	049	075	.384	676	23
38	078	104	.339	673	22
39	107	134	.295̄	671	21
40	.08136	.08163	12.251	.99668	**20**
41	165̄	192	.207	666	19
42	194	221	.163	664	18
43	223	251	.120	661	17
44	252	280	.077	659	16
45	.08281	.08309	12.035̄	.99657	**15**
46	310	339	11.992	654	14
47	339	368	.950	652	13
48	368	397	.909	649	12
49	397	427	.867	647	11
50	.08426	.08456	11.826	.99644	**10**
51	455̄	485	.785	642	9
52	484	514	.745̄	639	8
53	513	544	.705̄	637	7
54	542	573	.664	635̄	6
55	.08571	.08602	11.625̄	.99632	**5**
56	600	632	.585	630	4
57	629	661	.546	627	3
58	658	690	.507	625̄	2
59	687	720	.468	622	1
60	.08716	.08749	11.430	.99619	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.08716	.08749	11.430	.99619	**60**
1	745̄	778	.392	617	59
2	774	807	.354	614	58
3	803	837	.316	612	57
4	831	866	.279	609	56
5	.08860	.08895	11.242	.99607	**55**
6	889	925̄	.205̄	604	54
7	918	954	.168	602	53
8	947	.08983	.132	599	52
9	.08976	.09013	.095	596	51
10	.09005	.09042	11.059	.99594	**50**
11	034	071	11.024	591	49
12	063	101	10.988	588	48
13	092	130	.953	586	47
14	121	159	.918	583	46
15	.09150	.09189	10.883	.99580	**45**
16	179	218	.848	578	44
17	208	247	.814	575	43
18	237	277	.780	572	42
19	266	306	.746	570	41
20	.09295̄	.09335	10.712	.99567	**40**
21	324	365̄	.678	564	39
22	353	394	.645̄	562	38
23	382	423	.612	559	37
24	411	453	.579	556	36
25	.09440	.09482	10.546	.99553	**35**
26	469	511	.514	551	34
27	498	541	.481	548	33
28	527	570	.449	545	32
29	556	600	.417	542	31
30	.09585̄	.09629	10.385	.99540	**30**
31	614	658	.354	537	29
32	642	688	.322	534	28
33	671	717	.291	531	27
34	700	746	.260	528	26
35	.09729	.09776	10.229	.99526	**25**
36	758	805	.199	523	24
37	787	834	.168	520	23
38	816	864	.138	517	22
39	845	893	.108	514	21
40	09874	.09923	10.078	.99511	**20**
41	903	952	.048	508	19
42	932	.09981	10 019	506	18
43	961	.10011	9.9893	503	17
44	09990	040	.9601	5̄00	16
45	.10019	.10069	9.9310	.99497	**15**
46	048	099	.9021	494	14
47	077	128	.8734	491	13
48	106	158	.8448	488	12
49	135̄	187	.8164	485	11
50	.10164	.10216	9.7882	.99482	**10**
51	192	246	.7601	479	9
52	221	275	.7322	476	8
53	250	305̄	.7044	473	7
54	279	334	.6768	470	6
55	.10308	.10363	9.6493	.99467	**5**
56	337	393	.6220	464	4
57	366	422	.5949	461	3
58	395̄	452	.5679	458	2
59	424	481	.5411	455	1
60	.10453	.10510	9.5144	.99452	**0**
	N Cos	N Cot	N Tan	N Sin	′

85°—Natural Functions—84°

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'	N Sin	N Tan	N Cot	N Cos	'
0	.10453	.10510	9.5144	.99452	**60**
1	482	540	.4878	449	59
2	511	569	.4614	446	58
3	540	599	.4352	443	57
4	569	628	.4090	440	56
5	.10597	.10657	9.3831	.99437	**55**
6	626	687	.3572	434	54
7	655	716	.3315	431	53
8	684	746	.3060	428	52
9	713	775	.2806	424	51
10	.10742	.10805	9.2553	.99421	**50**
11	771	834	.2302	418	49
12	800	863	.2052	415	48
13	829	893	.1803	412	47
14	858	922	.1555	409	46
15	.10887	.10952	9.1309	.99406	**45**
16	916	.10981	.1065	402	44
17	945	.11011	.0821	399	43
18	.10973	040	.0579	396	42
19	.11002	070	.0338	393	41
20	.11031	.11099	9.0098	.99390	**40**
21	060	128	8.9860	386	39
22	089	158	.9623	383	38
23	118	187	.9387	380	37
24	147	217	.9152	377	36
25	.11176	.11246	8.8919	.99374	**35**
26	205	276	.8686	370	34
27	234	305	.8455	367	33
28	263	335	.8225	364	32
29	291	364	.7996	360	31
30	.11320	.11394	8.7769	.99357	**30**
31	349	423	.7542	354	29
32	378	452	.7317	351	28
33	407	482	.7093	347	27
34	436	511	.6870	344	26
35	.11465	.11541	8.6648	99341	**25**
36	494	570	.6427	337	24
37	523	600	.6208	334	23
38	552	629	.5989	331	22
39	580	659	.5772	327	21
40	.11609	.11688	8.5555	99324	**20**
41	638	718	.5340	320	19
42	667	747	.5126	317	18
43	696	777	.4913	314	17
44	725	806	.4701	310	16
45	.11754	.11836	8.4490	.99307	**15**
46	783	865	.4280	303	14
47	812	895	.4071	300	13
48	840	924	.3863	297	12
49	869	954	.3656	293	11
50	.11898	.11983	8.3450	.99290	**10**
51	927	.12013	.3245	286	9
52	956	042	.3041	283	8
53	.11985	072	.2838	279	7
54	.12014	101	.2636	276	6
55	.12043	.12131	8.2434	.99272	**5**
56	071	160	.2234	269	4
57	100	190	.2035	265	3
58	129	219	.1837	262	2
59	158	249	.1640	258	1
60	.12187	.12278	8.1443	.99255	**0**
	N Cos	N Cot	N Tan	N Sin	'

'	N Sin	N Tan	N Cot	N Cos	'
0	.12187	.12278	8.1443	.99255	**60**
1	216	308	.1248	251	59
2	245	338	.1054	248	58
3	274	367	.0860	244	57
4	302	397	.0667	240	56
5	.12331	.12426	8.0476	.99237	**55**
6	360	456	.0285	233	54
7	389	485	8.0095	230	53
8	418	515	7.9906	226	52
9	447	544	.9718	222	51
10	.12476	.12574	7.9530	.99219	**50**
11	504	603	.9344	215	49
12	533	633	.9158	211	48
13	562	662	.8973	208	47
14	591	692	.8789	204	46
15	.12620	.12722	7.8606	.99200	**45**
16	649	751	.8424	197	44
17	678	781	.8243	193	43
18	706	810	8062	189	42
19	735	840	.7882	186	41
20	.12764	.12869	7.7704	.99182	**40**
21	793	899	.7525	178	39
22	822	929	.7348	175	38
23	851	958	.7171	171	37
24	880	.12988	.6996	167	36
25	.12908	.13017	7.6821	.99163	**35**
26	937	047	.6647	160	34
27	966	076	.6473	156	33
28	.12995	106	.6301	152	32
29	.13024	136	.6129	148	31
30	.13053	.13165	7.5958	.99144	**30**
31	081	195	.5787	141	29
32	110	224	.5618	137	28
33	139	254	.5449	133	27
34	168	284	.5281	129	26
35	.13197	.13313	7.5113	.99125	**25**
36	226	343	.4947	122	24
37	254	372	.4781	118	23
38	283	402	.4615	114	22
39	312	432	.4451	110	21
40	.13341	.13461	7.4287	.99106	**20**
41	370	491	.4124	102	19
42	399	521	.3962	098	18
43	427	550	.3800	094	17
44	456	580	.3639	091	16
45	.13485	.13609	7.3479	.99087	**15**
46	514	639	.3319	083	14
47	543	669	.3160	079	13
48	572	698	.3002	075	12
49	600	728	.2844	071	11
50	.13629	.13758	7.2687	.99067	**10**
51	658	787	.2531	063	9
52	687	817	.2375	059	8
53	716	846	.2220	055	7
54	744	876	.2066	051	6
55	.13773	.13906	7.1912	.99047	**5**
56	802	935	.1759	043	4
57	831	965	.1607	039	3
58	860	.13995	.1455	035	2
59	889	.14024	.1304	031	1
60	.13917	.14054	7.1154	.99027	**0**
	N Cos	N Cot	N Tan	N Sin	'

83°—Natural Functions—82°

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8°—Natural Functions—9°

′	N Sin	N Tan	N Cot	N Cos	
0	.13917	.14054	7.1154	.99027	**60**
1	946	084	.1004	023	59
2	.13975̄	113	.0855̄	019	58
3	.14004	143	.0706	015̄	57
4	033	173	.0558	011	56
5	.14061	.14202	7.0410	.99006	**55**
6	090	232	.0264	.99002	54
7	119	262	7.0117	.98998	53
8	148	291	6.9972	994	52
9	177	321	.9827	990	51
10	.14205	.14351	6.9682	.98986	**50**
11	234	381	.9538	982	49
12	263	410	.9395	978	48
13	292	440	.9252	973	47
14	320	470	.9110	969	46
15	.14349	.14499	6.8969	.98965	**45**
16	378	529	.8828	961	44
17	407	559	.8687	957	43
18	436	588	.8548	953	42
19	464	618	.8408	948	41
20	.14493	.14648	6.8269	.98944	**40**
21	522	678	.8131	940	39
22	551	707	.7994	936	38
23	580	737	.7856	931	37
24	608	767	.7720	927	36
25	.14637	.14796	6.7584	.98923	**35**
26	666	826	.7448	919	34
27	695̄	856	.7313	914	33
28	723	886	.7179	910	32
29	752	915	.7045̄	906	31
30	.14781	.14945	6.6912	.98902	**30**
31	810	.14975̄	.6779	897	29
32	838	.15005̄	.6646	893	28
33	867	034	.6514	889	27
34	896	064	.6383	884	26
35	.14925̄	.15094	6.6252	.98880	**25**
36	954	124	.6122	876	24
37	.14982	153	.5992	871	23
38	.15011	183	.5863	867	22
39	040	213	.5734	863	21
40	.15069	.15243	6.5606	.98858	**20**
41	097	272	.5478	854	19
42	126	302	.5350	849	18
43	155̄	332	.5223	845̄	17
44	184	362	.5097	841	16
45	.15212	.15391	6.4971	.98836	**15**
46	241	421	.4846	832	14
47	270	451	.4721	827	13
48	299	481	.4596	823	12
49	327	511	.4472	818	11
50	.15356	.15540	6.4348	.98814	**10**
51	385̄	570	.4225	809	9
52	414	600	.4103	805̄	8
53	442	630	.3980	800	7
54	471	660	.3859	796	6
55	.15500	.15689	6.3737	.98791	**5**
56	529	719	.3617	787	4
57	557	749	.3496	782	3
58	586	779	.3376	778	2
59	615̄	809	.3257	773	1
60	.15643	.15838	6.3138	.98769	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.15643	.15838	6.3138	.98769	**60**
1	672	868	.3019	764	59
2	701	898	.2901	760	58
3	730	928	.2783	755	57
4	758	958	.2666	751	56
5	.15787	.15988	6.2549	.98746	**55**
6	816	.16017	.2432	741	54
7	845̄	047	.2316	737	53
8	873	077	.2200	732	52
9	902	107	.2085	728	51
10	.15931	.16137	6.1970	.98723	**50**
11	959	167	.1856	718	49
12	.15988	196	.1742	714	48
13	.16017	226	.1628	709	47
14	046	256	.1515	704	46
15	.16074	.16286	6.1402	.98700	**45**
16	103	316	.1290	695̄	44
17	132	346	.1178	690	43
18	160	376	.1066	686	42
19	189	405	.0955	681	41
20	.16218	.16435	6.0844	.98676	**40**
21	246	465	.0734	671	39
22	275	495	.0624	667	38
23	304	525	.0514	662	37
24	333	555̄	.0405	657	36
25	.16361	.16585̄	6.0296	.98652	**35**
26	390	615̄	.0188	648	34
27	419	645̄	6.0080	643	33
28	447	674	5.9972	638	32
29	476	704	.9865̄	633	31
30	.16505̄	.16734	5.9758	.98629	**30**
31	533	764	.9651	624	29
32	562	794	.9545̄	619	28
33	591	824	.9439	614	27
34	620	854	.9333	609	26
35	.16648	.16884	5.9228	.98604	**25**
36	677	914	.9124	600	24
37	706	944	.9019	595̄	23
38	734	.16974	.8915	590	22
39	763	.17004	.8811	585	21
40	.16792	.17033	5.8708	.98580	**20**
41	820	063	.8605	575	19
42	849	093	.8502	570	18
43	878	123	.8400	565	17
44	906	153	.8298	561	16
45	.16935̄	.17183	5.8197	.98556	**15**
46	964	213	.8095	551	14
47	.16992	243	.7994	546	13
48	.17021	273	.7894	541	12
49	050	303	.7794	536	11
50	.17078	.17333	5.7694	.98531	**10**
51	107	363	.7594	526	9
52	136	393	.7495̄	521	8
53	164	423	.7396	516	7
54	193	453	.7297	511	6
55	.17222	.17483	5.7199	.98506	**5**
56	250	513	.7101	501	4
57	279	543	.7004	496	3
58	308	573	.6906	491	2
59	336	603	.6809	486	1
60	.17365̄	.17633	5.6713	.98481	**0**
	N Cos	N Cot	N Tan	N Sin	′

81°—Natural Functions—80°

'	N Sin	N Tan	N Cot	N Cos	
0	.17365̄	.17633	5.6713	.98481	60
1	393	663	.6617	476	59
2	422	693	.6521	471	58
3	451	723	.6425̄	466	57
4	479	753	.6329	461	56
5	.17508	.17783	5.6234	.98455	55
6	537	813	.6140	450	54
7	565	843	.6045	445	53
8	594	873	.5951	440	52
9	623	903	.5857	435̄	51
10	.17651	.17933	5.5764	.98430	50
11	680	963	.5671	425̄	49
12	708	.17993	.5578	420	48
13	737	.18023	.5485	414	47
14	766	053	.5393	409	46
15	.17794	.18083	5.5301	.98404	45
16	823	113	.5209	399	44
17	852	143	.5118	394	43
18	880	173	.5026	389	42
19	909	203	.4936	383	41
20	.17937	.18233	5.4845	.98378	40
21	966	263	.4755̄	373	39
22	.17995̄	293	.4665̄	368	38
23	.18023	323	.4575	362	37
24	052	353	.4486	357	36
25	.18081	.18384	5.4397	.98352	35
26	109	414	.4308	347	34
27	138	444	.4219	341	33
28	166	474	.4131	336	32
29	195̄	504	.4043	331	31
30	.18224	.18534	5.3955	.98325	30
31	252	564	.3868	320	29
32	281	594	.3781	315̄	28
33	309	624	.3694	310	27
34	338	654	.3607	304	26
35	.18367	.18684	5.3521	.98299	25
36	395	714	.3435̄	294	24
37	424	745̄	.3349	288	23
38	452	775̄	.3263	283	22
39	481	805̄	.3178	277	21
40	.18509	.18835̄	5.3093	.98272	20
41	538	865	.3008	267	19
42	567	895	.2924	261	18
43	595	925	.2839	256	17
44	624	955	.2755	250	16
45	.18652	.18986	5.2672	.98245	15
46	681	.19016	.2588	240	14
47	710	046	.2505̄	234	13
48	738	076	.2422	229	12
49	767	106	.2339	223	11
50	.18795	.19136	5.2257	.98218	10
51	824	166	.2174	212	9
52	852	197	.2092	207	8
53	881	227	.2011	201	7
54	910	257	.1929	196	6
55	.18938	.19287	5.1848	.98190	5
56	967	317	.1767	185̄	4
57	.18995	347	.1686	179	3
58	.19024	378	.1606	174	2
59	052	408	.1526	168	1
60	.19081	.19438	5.1446	.98163	0
	N Cos	N Cot	N Tan	N Sin	'

'	N Sin	N Tan	N Cot	N Cos	
0	.19081	.19438	5.1446	.98163	60
1	109	468	.1366	157	59
2	138	498	.1286	152	58
3	167	529	.1207	146	57
4	195	559	.1128	140	56
5	.19224	.19589	5.1049	.98135̄	55
6	252	619	.0970	129	54
7	281	649	.0892	124	53
8	309	680	.0814	118	52
9	338	710	.0736	112	51
10	.19366	.19740	5.0658	.98107	50
11	395̄	770	.0581	101	49
12	423	801	.0504	096	48
13	452	831	.0427	090	47
14	481	861	.035̄0	084	46
15	.19509	.19891	5.0273	.98079	45
16	538	921	.0197	073	44
17	566	952	.0121	067	43
18	595̄	.19982	5.0045	061	42
19	623	.20012	4.9969	056	41
20	.19652	.20042	4.9894	.98050	40
21	680	073	.9819	044	39
22	709	103	.9744	039	38
23	737	133	.9669	033	37
24	766	164	.9594	027	36
25	.19794	.20194	4.9520	.98021	35
26	823	224	.9446	016	34
27	851	254	.9372	010	33
28	880	285̄	.9298	.98004	32
29	908	315̄	.9225̄	.97998	31
30	.19937	.20345	4.9152	.97992	30
31	965	376	.9078	987	29
32	.19994	406	.9006	981	28
33	.20022	436	.8933	975	27
34	051	466	.8860	969	26
35	.20079	.20497	4.8788	.97963	25
36	108	527	.8716	958	24
37	136	557	.8644	952	23
38	165̄	588	.8573	946	22
39	193	618	.8501	940	21
40	.20222	.20648	4.8430	.97934	20
41	250	679	.8359	928	19
42	279	709	.8288	922	18
43	307	739	.8218	916	17
44	336	770	.8147	910	16
45	.20364	.20800	4.8077	.97905̄	15
46	393	830	.8007	899	14
47	421	861	.7937	893	13
48	45̄0	891	.7867	887	12
49	478	921	.7798	881	11
50	.20507	.20952	4.7729	.97875̄	10
51	535	.20982	.7659	869	9
52	563	.21013	.7591	863	8
53	592	043	.7522	857	7
54	620	073	.7453	851	6
55	.20649	.21104	4.7385	.97845̄	5
56	677	134	.7317	839	4
57	706	164	.7249	833	3
58	734	195̄	.7181	827	2
59	763	225	.7114	821	1
60	.20791	.21256	4.7046	.97815̄	0
	N Cos	N Cot	N Tan	N Sin	'

79°—Natural Functions—78°

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'	N Sin	N Tan	N Cot	N Cos	
0	.20791	.21256	4.7046	.97815	60
1	820	286	.6979	809	59
2	848	316	.6912	803	58
3	877	347	.6845	797	57
4	905	377	.6779	791	56
5	.20933	.21408	4.6712	.97784	55
6	962	438	.6646	778	54
7	.20990	469	.6580	772	53
8	.21019	499	.6514	766	52
9	047	529	.6448	760	51
10	.21076	.21560	4.6382	.97754	50
11	104	590	.6317	748	49
12	132	621	.6252	742	48
13	161	651	.6187	735	47
14	189	682	.6122	729	46
15	.21218	.21712	4.6057	.97723	45
16	246	743	.5993	717	44
17	275	773	.5928	711	43
18	303	804	.5864	705	42
19	331	834	.5800	698	41
20	.21360	.21864	4.5736	.97692	40
21	388	895	.5673	686	39
22	417	925	.5609	680	38
23	445	956	.5546	673	37
24	474	.21986	.5483	667	36
25	.21502	.22017	4.5420	.97661	35
26	530	047	.5357	655	34
27	559	078	.5294	648	33
28	587	108	.5232	642	32
29	616	139	.5169	636	31
30	.21644	.22169	4.5107	.97630	30
31	672	200	.5045	623	29
32	701	231	.4983	617	28
33	729	261	.4922	611	27
34	758	292	.4860	604	26
35	.21786	.22322	4.4799	.97598	25
36	814	353	.4737	592	24
37	843	383	.4676	585	23
38	871	414	.4615	579	22
39	899	444	.4555	573	21
40	.21928	.22475	4.4494	.97566	20
41	956	505	.4434	560	19
42	.21985	536	.4373	553	18
43	.22013	567	.4313	547	17
44	041	597	.4253	541	16
45	.22070	.22628	4.4194	.97534	15
46	098	658	.4134	528	14
47	126	689	.4075	521	13
48	155	719	.4015	515	12
49	183	750	.3956	508	11
50	.22212	.22781	4.3897	.97502	10
51	240	811	.3838	496	9
52	268	842	.3779	489	8
53	297	872	.3721	483	7
54	325	903	.3662	476	6
55	.22353	.22934	4.3604	.97470	5
56	382	964	.3546	463	4
57	410	.22995	.3488	457	3
58	438	.23026	.3430	450	2
59	467	056	.3372	444	1
60	.22495	.23087	4.3315	.97437	0
	N Cos	N Cot	N Tan	N Sin	'

'	N Sin	N Tan	N Cot	N Cos	
0	.22495	.23087	4.3315	.97437	60
1	523	117	.3257	430	59
2	552	148	.3200	424	58
3	580	179	.3143	417	57
4	608	209	.3086	411	56
5	.22637	.23240	4.3029	.97404	55
6	665	271	.2972	398	54
7	693	301	.2916	391	53
8	722	332	.2859	384	52
9	750	363	.2803	378	51
10	.22778	.23393	4.2747	.97371	50
11	807	424	.2691	365	49
12	835	455	.2635	358	48
13	863	485	.2580	351	47
14	892	516	.2524	345	46
15	.22920	.23547	4.2468	.97338	45
16	948	578	.2413	331	44
17	.22977	608	.2358	325	43
18	.23005	639	.2303	318	42
19	033	670	.2248	311	41
20	.23062	.23700	4.2193	.97304	40
21	090	731	.2139	298	39
22	118	762	.2084	291	38
23	146	793	.2030	284	37
24	175	823	.1976	278	36
25	.23203	.23854	4.1922	.97271	35
26	231	885	.1868	264	34
27	260	916	.1814	257	33
28	288	946	.1760	251	32
29	316	.23977	.1706	244	31
30	23345	.24008	4.1653	.97237	30
31	373	039	.1600	230	29
32	401	069	.1547	223	28
33	429	100	.1493	217	27
34	458	131	.1441	210	26
35	23486	.24162	4.1388	.97203	25
36	514	193	.1335	196	24
37	542	223	.1282	189	23
38	571	254	.1230	182	22
39	599	285	.1178	176	21
40	.23627	.24316	4.1126	.97169	20
41	656	347	.1074	162	19
42	684	377	.1022	155	18
43	712	408	.0970	148	17
44	740	439	.0918	141	16
45	.23769	.24470	4.0867	.97134	15
46	797	501	.0815	127	14
47	825	532	.0764	120	13
48	853	562	.0713	113	12
49	882	593	.0662	106	11
50	.23910	.24624	4.0611	.97100	10
51	938	655	.0560	093	9
52	966	686	.0509	086	8
53	.23995	717	.0459	079	7
54	.24023	747	.0408	072	6
55	.24051	.24778	4.0358	.97065	5
56	079	809	.0308	058	4
57	108	840	.0257	051	3
58	136	871	.0207	044	2
59	164	902	.0158	037	1
60	.24192	.24933	4.0108	.97030	0
	N Cos	N Cot	N Tan	N Sin	'

′	N Sin	N Tan	N Cot	N Cos	
0	.24192	.24933	4.0108	.97030	**60**
1	220	964	.0058	023	59
2	249	.24995̄	4.0009	015	58
3	277	.25026	3.9959	008	57
4	305	056	.9910	.97001	56
5	.24333	.25087	3.9861	.96994	**55**
6	362	118	.9812	987	54
7	390	149	.9763	980	53
8	418	180	.9714	973	52
9	446	211	.9665	966	51
10	.24474	.25242	3.9617	.96959	**50**
11	503	273	.9568	952	49
12	531	304	.9520	945̄	48
13	559	335̄	.9471	937	47
14	587	366	.9423	930	46
15	.24615	.25397	3.9375	.96923	**45**
16	644	428	.9327	916	44
17	672	459	.9279	909	43
18	700	490	.9232	902	42
19	728	521	.9184	894	41
20	.24756	.25552	3.9136	.96887	**40**
21	784	583	.9089	880	39
22	813	614	.9042	873	38
23	841	645̄	.8995̄	866	37
24	869	676	.8947	858	36
25	.24897	.25707	3.8900	.96851	**35**
26	925	738	.8854	844	34
27	954	769	.8807	837	33
28	.24982	800	.8760	829	32
29	.25010	831	.8714	822	31
30	.25038	.25862	3.8667	.96815̄	**30**
31	066	893	.8621	807	29
32	094	924	.8575̄	800	28
33	122	955̄	.8528	793	27
34	151	.25986	.8482	786	26
35	.25179	.26017	3.8436	.96778	**25**
36	207	048	.8391	771	24
37	235	079	.8345̄	764	23
38	263	110	.8299	756	22
39	291	141	.8254	749	21
40	.25320	.26172	3.8208	.96742	**20**
41	348	203	.8163	734	19
42	376	235̄	.8118	727	18
43	404	266	.8073	719	17
44	432	297	.8028	712	16
45	.25460	.26328	3.7983	.96705̄	**15**
46	488	359	.7938	697	14
47	516	390	.7893	690	13
48	545̄	421	.7848	682	12
49	573	452	.7804	675̄	11
50	.25601	.26483	3.7760	.96667	**10**
51	629	515̄	.7715	660	9
52	657	546	.7671	653	8
53	685	577	.7627	645	7
54	713	608	.7583	638	6
55	.25741	.26639	3.7539	.96630	**5**
56	769	670	.7495̄	623	4
57	798	701	.7451	615	3
58	826	733	.7408	608	2
59	854	764	.7364	600	1
60	.25882	.26795̄	3.7321	.96593	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.25882	.26795̄	3.7321	.96593	**60**
1	910	826	.7277	585	59
2	938	857	.7234	578	58
3	966	888	.7191	570	57
4	.25994	920	.7148	562	56
5	.26022	.26951	3.7105̄	.96555̄	**55**
6	050	.26982	.7062	547	54
7	079	.27013	.7019	540	53
8	107	044	.6976	532	52
9	135̄	076	.6933	524	51
10	.26163	.27107	3.6891	.96517	**50**
11	191	138	.6848	509	49
12	219	169	.6806	502	48
13	247	201	.6764	494	47
14	275	232	.6722	486	46
15	.26303	.27263	3.6680	.96479	**45**
16	331	294	.6638	471	44
17	359	326	.6596	463	43
18	387	357	.6554	456	42
19	415	388	.6512	448	41
20	.26443	.27419	3.6470	.96440	**40**
21	471	451	.6429	433	39
22	500	482	.6387	425̄	38
23	528	513	.6346	417	37
24	556	545̄	.6305̄	410	36
25	.26584	.27576	3.6264	.96402	**35**
26	612	607	.6222	394	34
27	640	638	.6181	386	33
28	668	670	.6140	379	32
29	696	701	.6100	371	31
30	.26724	.27732	3.6059	.96363	**30**
31	752	764	.6018	355	29
32	780	795	.5978	347	28
33	808	826	.5937	340	27
34	836	858	.5897	332	26
35	.26864	.27889	3.5856	.96324	**25**
36	892	921	.5816	316	24
37	920	952	.5776	308	23
38	948	.27983	.5736	301	22
39	.26976	.28015̄	.5696	293	21
40	.27004	.28046	3.5656	.96285̄	**20**
41	032	077	.5616	277	19
42	060	109	.5576	269	18
43	088	140	.5536	261	17
44	116	172	.5497	253	16
45	.27144	.28203	3.5457	.96246	**15**
46	172	234	.5418	238	14
47	200	266	.5379	230	13
48	228	297	.5339	222	12
49	256	329	.5300	214	11
50	.27284	.28360	3.5261	.96206	**10**
51	312	391	.5222	198	9
52	340	423	.5183	190	8
53	368	454	.5144	182	7
54	396	486	.5105	174	6
55	.27424	.28517	3.5067	.96166	**5**
56	452	549	.5028	158	4
57	480	580	.4989	150	3
58	508	612	.4951	142	2
59	536	643	.4912	134	1
60	.27564	.28675̄	3.4874	.96126	**0**
	N Cos	N Cot	N Tan	N Sin	′

75°—Natural Functions—74°

′	N Sin	N Tan	N Cot	N Cos	
0	.27564	.28675̄	3.4874	.96126	**60**
1	592	706	.4836	118	59
2	620	738	.4798	110	58
3	648	769	.4760	102	57
4	676	801	.4722	094	56
5	.27704	.28832	3.4684	.96086	**55**
6	731	864	.4646	078	54
7	759	895	.4608	070	53
8	787	927	.4570	062	52
9	815	958	.4533	054	51
10	.27843	.28990	3.4495	.96046	**50**
11	871	.29021	.4458	037	49
12	899	053	.4420	029	48
13	927	084	.4383	021	47
14	955̄	116	.4346	013	46
15	.27983	.29147	3.4308	.96005̄	**45**
16	.28011	179	.4271	.95997	44
17	039	210	.4234	989	43
18	067	242	.4197	981	42
19	095̄	274	.4160	972	41
20	.28123	.29305	3.4124	.95964	**40**
21	150	337	.4087	956	39
22	178	368	.4050	948	38
23	206	400	.4014	940	37
24	234	432	.3977	931	36
25	.28262	.29463	3.3941	.95923	**35**
26	290	495̄	.3904	915̄	34
27	318	526	.3868	907	33
28	346	558	.3832	898	32
29	374	590	.3796	890	31
30	.28402	.29621	3.3759	.95882	**30**
31	429	653	.3723	874	29
32	457	685̄	.3687	865	28
33	485	716	.3652	857	27
34	513	748	.3616	849	26
35	.28541	.29780	3.3580	.95841	**25**
36	569	811	.3544	832	24
37	597	843	.3509	824	23
38	625̄	875̄	.3473	816	22
39	652	906	.3438	807	21
40	.28680	.29938	3.3402	.95799	**20**
41	708	.29970	.3367	791	19
42	736	.30001	.3332	782	18
43	764	033	.3297	774	17
44	792	065̄	.3261	766	16
45	.28820	.30097	3.3226	.95757	**15**
46	847	128	.3191	749	14
47	875	160	.3156	740	13
48	903	192	.3122	732	12
49	931	224	.3087	724	11
50	.28959	.30255	3.3052	.95715	**10**
51	.28987	287	.3017	707	9
52	.29015̄	319	.2983	698	8
53	042	351	.2948	690	7
54	070	382	.2914	681	6
55	.29098	.30414	3.2879	.95673	**5**
56	126	446	.2845	664	4
57	154	478	.2811	656	3
58	182	509	.2777	647	2
59	209	541	.2743	639	1
60	.29237	.30573	3.2709	.95630	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.29237	.30573	3.2709	.95630	**60**
1	265̄	605̄	.2675̄	622	59
2	293	637	.2641	613	58
3	321	669	.2607	605̄	57
4	348	700	.2573	596	56
5	.29376	.30732	3.2539	.95588	**55**
6	404	764	.2506	579	54
7	432	796	.2472	571	53
8	460	828	.2438	562	52
9	487	860	.2405̄	554	51
10	.29515	.30891	3.2371	.95545	**50**
11	543	923	.2338	536	49
12	571	955	.2305̄	528	48
13	599	.30987	.2272	519	47
14	626	.31019	.2238	511	46
15	.29654	.31051	3.2205	.95502	**45**
16	682	083	.2172	493	44
17	710	115̄	.2139	485̄	43
18	737	147	.2106	476	42
19	765	178	.2073	467	41
20	.29793	.31210	3.2041	.95459	**40**
21	821	242	.2008	450	39
22	849	274	.1975	441	38
23	876	306	.1943	433	37
24	904	338	.1910	424	36
25	.29932	.31370	3.1878	.95415	**35**
26	960	402	.1845	407	34
27	.29987	434	.1813	398	33
28	.30015	466	.1780	389	32
29	043	498	.1748	380	31
30	.30071	.31530	3.1716	.95372	**30**
31	098	562	.1684	363	29
32	126	594	.1652	354	28
33	154	626	.1620	345	27
34	182	658	.1588	337	26
35	.30209	.31690	3.1556	.95328	**25**
36	237	722	.1524	319	24
37	265̄	754	.1492	310	23
38	292	786	.1460	301	22
39	320	818	.1429	293	21
40	.30348	.3185̄0	3.1397	.95284	**20**
41	376	882	.1366	275̄	19
42	403	914	.1334	266	18
43	431	946	.1303	257	17
44	459	.31978	.1271	248	16
45	.30486	.32010	3.1240	.95240	**15**
46	514	042	.1209	231	14
47	542	074	.1178	222	13
48	570	106	.1146	213	12
49	597	139	.1115	204	11
50	.30625̄	.32171	3.1084	.95195	**10**
51	653	203	.1053	186	9
52	680	235̄	.1022	177	8
53	708	267	.0991	168	7
54	736	299	.0961	159	6
55	.30763	.32331	3.0930	.95150	**5**
56	791	363	.0899	142	4
57	819	396	.0868	133	3
58	846	428	.0838	124	2
59	874	460	.0807	115̄	1
60	.30902	.32492	3.0777	.95106	**0**
	N Cos	N Cot	N Tan	N Sin	′

73°—Natural Functions—72°

′	N Sin	N Tan	N Cot	N Cos	
0	.30902	.32492	3.0777	.95106	60
1	929	524	.0746	097	59
2	957	556	.0716	088	58
3	.30985	588	.0686	079	57
4	.31012	621	.0655	070	56
5	.31040	.32653	3.0625	.95061	55
6	068	685	.0595	052	44
7	095	717	.0565	043	53
8	123	749	.0535	033	52
9	151	782	.0505	024	51
10	.31178	.32814	3.0475	.95015	50
11	[redacted]				49
12	233	878	.0415	.94997	48
13	261	911	.0385	988	47
14	289	943	.0356	979	46
15	.31316	.32975	3.0326	.94970	45
16	344	.33007	.0296	961	44
17	372	040	.0267	952	43
18	399	072	.0237	943	42
19	427	104	.0208	933	41
20	.31454	.33136	3.0178	.94924	40
21	482	169	.0149	915	39
22	510	201	.0120	906	38
23	537	233	.0090	897	37
24	565	266	.0061	888	36
25	.31593	.33298	3.0032	.94878	35
26	620	330	3.0003	869	34
27	648	363	2.9974	860	33
28	675	395	.9945	851	32
29	703	427	.9916	842	31
30	.31730	.33460	2.9887	.94832	30
31	758	492	.9858	823	29
32	786	524	.9829	814	28
33	813	557	.9800	805	27
34	841	589	.9772	795	26
35	.31868	.33621	2.9743	.94786	25
36	896	654	.9714	777	24
37	923	686	.9686	768	23
38	951	718	.9657	758	22
39	.31979	751	.9629	749	21
40	.32006	.33783	2.9600	.94740	20
41	034	816	.9572	730	19
42	061	848	.9544	721	18
43	089	881	.9515	712	17
44	116	913	.9487	702	16
45	.32144	.33945	2.9459	.94693	15
46	171	.33978	.9431	684	14
47	199	.34010	.9403	674	13
48	227	043	.9375	665	12
49	254	075	.9347	656	11
50	.32282	.34108	2.9319	.94646	10
51	309	140	.9291	637	9
52	337	173	.9263	627	8
53	364	205	.9255	618	7
54	392	238	.9208	609	6
55	.32419	.34270	2.9180	.94599	5
56	447	303	.9152	590	4
57	474	335	.9125	580	3
58	502	368	.9097	571	2
59	529	400	.9070	561	1
60	.32557	.34433	2.9042	.94552	0
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.32557	.34433	2.9042	.94552	60
1	584	465	.9015	542	59
2	612	498	.8987	533	58
3	639	530	.8960	523	57
4	667	563	.8933	514	56
5	.32694	.34596	2.8905	.94504	55
6	722	628	.8878	495	54
7	749	661	.8851	485	53
8	777	693	.8824	476	52
9	804	726	.8797	466	51
10	.32832	.34758	2.8770	.94457	50
11	859	791	.8743	447	49
12	887	824	.8716	438	48
13	914	856	.8689	428	47
14	942	889	.8662	418	46
15	.32969	.34922	2.8636	.94409	45
16	.32997	954	.8609	399	44
17	.33024	.34987	.8582	390	43
18	051	.35020	.8556	380	42
19	079	052	.8529	370	41
20	.33106	.35085	2.8502	.94361	40
21	134	118	.8476	351	39
22	161	150	.8449	342	38
23	189	183	.8423	332	37
24	216	216	.8397	322	36
25	.33244	.35248	2.8370	.94313	35
26	271	281	.8344	303	34
27	298	314	.8318	293	33
28	326	346	.8291	284	32
29	353	379	.8265	274	31
30	.33381	.35412	2.8239	.94264	30
31	408	445	.8213	254	29
32	436	477	.8187	245	28
33	463	510	.8161	235	27
34	490	543	.8135	225	26
35	.33518	.35576	2.8109	.94215	25
36	545	608	.8083	206	24
37	573	641	.8057	196	23
38	600	674	.8032	186	22
39	627	707	.8006	176	21
40	.33655	.35740	2.7980	.94167	20
41	682	772	.7955	157	19
42	710	805	.7929	147	18
43	737	838	.7903	137	17
44	764	871	.7878	127	16
45	.33792	.35904	2.7852	.94118	15
46	819	937	.7827	108	14
47	846	.35969	.7801	098	13
48	874	.36002	.7776	088	12
49	901	035	.7751	078	11
50	.33929	.36068	2.7725	.94068	10
51	956	101	.7700	058	9
52	.33983	134	.7675	049	8
53	.34011	167	.7650	039	7
54	038	199	.7625	029	6
55	.34065	.36232	2.7600	.94019	5
56	093	265	.7575	.94009	4
57	120	298	.7550	.93999	3
58	147	331	.7525	989	2
59	175	364	.7500	979	1
60	.34202	.36397	2.7475	.93969	0
	N Cos	N Cot	N Tan	N Sin	′

71°—Natural Functions—70°

20°—Natural Functions—21°

′	N Sin	N Tan	N Cot	N Cos	
0	.34202	.36397	2.7475	.93969	**60**
1	229	430	.7450	959	59
2	257	463	.7425	949	58
3	284	496	.7400	939	57
4	311	529	.7376	929	56
5	.34339	.36562	2.7351	.93919	**55**
6	366	595	.7326	909	54
7	393	628	.7302	899	53
8	421	661	.7277	889	52
9	448	694	.7253	879	51
10	.34475	.36727	2.7228	.93869	**50**
11	503	760	.7204	859	49
12	530	793	.7179	849	48
13	557	826	.7155	839	47
14	584	859	.7130	829	46
15	.34612	.36892	2.7106	.93819	**45**
16	639	925	.7082	809	44
17	666	958	.7058	799	43
18	694	.36991	.7034	789	42
19	721	.37024	.7009	779	41
20	.34748	.37057	2.6985	.93769	**40**
21	775	090	.6961	759	39
22	803	123	.6937	748	38
23	830	157	.6913	738	37
24	857	190	.6889	728	36
25	.34884	.37223	2.6865	.93718	**35**
26	912	256	.6841	708	34
27	939	289	.6818	698	33
28	966	322	.6794	688	32
29	.34993	355	.6770	677	31
30	.35021	.37388	2.6746	.93667	**30**
31	048	422	.6723	657	29
32	075	455	.6699	647	28
33	102	488	.6675	637	27
34	130	521	.6652	626	26
35	.35157	.37554	2.6628	.93616	**25**
36	184	588	.6605	606	24
37	211	621	.6581	596	23
38	239	654	.6558	585	22
39	266	687	.6534	575	21
40	.35293	.37720	2.6511	.93565	**20**
41	320	754	.6488	555	19
42	347	787	.6464	544	18
43	375	820	.6441	534	17
44	402	853	.6418	524	16
45	.35429	.37887	2.6395	.93514	**15**
46	456	920	.6371	503	14
47	484	953	.6348	493	13
48	511	.37986	.6325	483	12
49	538	.38020	.6302	472	11
50	.35565	.38053	2.6279	.93462	**10**
51	592	086	.6256	452	9
52	619	120	.6233	441	8
53	647	153	.6210	431	7
54	674	186	.6187	420	6
55	.35701	.38220	2.6165	.93410	**5**
56	728	253	.6142	400	4
57	755	286	.6119	389	3
58	782	320	.6096	379	2
59	810	353	.6074	368	1
60	.35837	.38386	2.6051	.93358	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.35837	.38386	2.6051	.93358	**60**
1	864	420	.6028	348	59
2	891	453	.6006	337	58
3	918	487	.5983	327	57
4	945	520	.5961	316	56
5	.35973	.38553	2.5938	.93306	**55**
6	.36000	587	.5916	295	54
7	027	620	.5893	285	53
8	054	654	.5871	274	52
9	081	687	.5848	264	51
10	.36108	.38721	2.5826	.93253	**50**
11	135	754	.5804	243	49
12	162	787	.5782	232	48
13	190	821	.5759	222	47
14	217	854	.5737	211	46
15	.36244	.38888	2.5715	.93201	**45**
16	271	921	.5693	190	44
17	298	955	.5671	180	43
18	325	.38988	.5649	169	42
19	352	.39022	.5627	159	41
20	.36379	.39055	2.5605	.93148	**40**
21	406	089	.5583	137	39
22	434	122	.5561	127	38
23	461	156	.5539	116	37
24	488	190	.5517	106	36
25	.36515	.39223	2.5495	.93095	**35**
26	542	257	.5473	084	34
27	569	290	.5452	074	33
28	596	324	.5430	063	32
29	623	357	.5408	052	31
30	.36650	39391	2.5386	.93042	**30**
31	677	425	.5365	031	29
32	704	458	.5343	020	28
33	731	492	.5322	.93010	27
34	758	526	.5300	.92999	26
35	.36785	.39559	2.5279	.92988	**25**
36	812	593	.5257	978	24
37	839	626	.5236	967	23
38	867	660	.5214	956	22
39	894	694	.5193	945	21
40	.36921	.39727	2.5172	.92935	**20**
41	948	761	.5150	924	19
42	.36975	795	.5129	913	18
43	.37002	829	.5108	902	17
44	029	862	.5086	892	16
45	.37056	.39896	2.5065	.92881	**15**
46	083	930	.5044	870	14
47	110	963	.5023	859	13
48	137	.39997	.5002	849	12
49	164	.40031	.4981	838	11
50	.37191	.40065	2.4960	.92827	**10**
51	218	098	.4939	816	9
52	245	132	.4918	805	8
53	272	166	.4897	794	7
54	299	200	.4876	784	6
55	.37326	.40234	2.4855	.92773	**5**
56	353	267	.4834	762	4
57	380	301	.4813	751	3
58	407	335	.4792	740	2
59	434	369	.4772	729	1
60	.37461	.40403	2.4751	.92718	**0**
	N Cos	N Cot	N Tan	N Sin	′

69°—Natural Functions—68°

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′	N Sin	N Tan	N Cot	N Cos	
0	.37461	.40403	2.4751	.92718	**60**
1	488	436	.4730	707	59
2	515	470	.4709	697	58
3	542	504	.4689	686	57
4	569	538	.4668	675	56
5	.37595	.40572	2.4648	.92664	**55**
6	622	606	.4627	653	54
7	649	640	.4606	642	53
8	676	674	.4586	631	52
9	703	707	.4566	620	51
10	.37730	.40741	2.4545	.92609	**50**
11	757	775	.4525	598	49
12	784	809	.4504	587	48
13	811	843	.4484	576	47
14	838	877	.4464	565	46
15	.37865	.40911	2.4443	.92554	**45**
16	892	945	.4423	543	44
17	919	.40979	.4403	532	43
18	946	.41013	.4383	521	42
19	973	047	.4362	510	41
20	.37999	.41081	2.4342	.92499	**40**
21	.38026	115	.4322	488	39
22	053	149	.4302	477	38
23	080	183	.4282	466	37
24	107	217	.4262	455	36
25	.38134	.41251	2.4242	.92444	**35**
26	161	285	.4222	432	34
27	188	319	.4202	421	33
28	215	353	.4182	410	32
29	241	387	.4162	399	31
30	.38268	.41421	2.4142	.92388	**30**
31	295	455	.4122	377	29
32	322	490	.4102	366	28
33	349	524	.4083	355	27
34	376	558	.4063	343	26
35	.38403	.41592	2.4043	.92332	**25**
36	430	626	.4023	321	24
37	456	660	.4004	310	23
38	483	694	.3984	299	22
39	510	728	.3964	287	21
40	.38537	.41763	2.3945	.92276	**20**
41	564	797	.3925	265	19
42	591	831	.3906	254	18
43	617	865	.3886	243	17
44	644	899	.3867	231	16
45	.38671	.41933	2.3847	.92220	**15**
46	698	.41968	.3828	209	14
47	725	.42002	.3808	198	13
48	752	036	.3789	186	12
49	778	070	.3770	175	11
50	.38805	.42105	2.3750	.92164	**10**
51	832	139	.3731	152	9
52	859	173	.3712	141	8
53	886	207	.3693	130	7
54	912	242	.3673	119	6
55	.38939	.42276	2.3654	.92107	**5**
56	966	310	.3635	096	4
57	.38993	345	.3616	085	3
58	.39020	379	.3597	073	2
59	046	413	.3578	062	1
60	.39073	.42447	2.3559	.92050	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.39073	.42447	2.3559	92050	**60**
1	100	482	3539	039	59
2	127	516	.3520	028	58
3	153	551	.3501	016	57
4	180	585	.3483	.92005	56
5	.39207	.42619	2.3464	.91994	**55**
6	234	654	.3445	982	54
7	260	688	.3426	971	53
8	287	722	.3407	959	52
9	314	757	.3388	948	51
10	.39341	.42791	2.3369	.91936	**50**
11	367	826	.3351	925	49
12	394	860	.3332	914	48
13	421	894	.3313	902	47
14	448	929	.3294	891	46
15	.39474	.42963	2.3276	.91879	**45**
16	501	.42998	.3257	868	44
17	528	.43032	.3238	856	43
18	555	067	.3220	845	42
19	581	101	.3201	833	41
20	.39608	.43136	2.3183	.91822	**40**
21	635	170	.3164	810	39
22	661	205	.3146	799	38
23	688	239	.3127	787	37
24	715	274	.3109	775	36
25	39741	.43308	2.3090	.91764	**35**
26	768	343	.3072	752	34
27	795	378	.3053	741	33
28	822	412	.3035	729	32
29	848	447	.3017	718	31
30	.39875	.43481	2.2998	.91706	**30**
31	902	516	.2980	694	29
32	928	550	.2962	683	28
33	955	585	.2944	671	27
34	.39982	620	.2925	660	26
35	.40008	.43654	2.2907	.91648	**25**
36	035	689	.2889	636	24
37	062	724	.2871	625	23
38	088	758	.2853	613	22
39	115	793	.2835	601	21
40	.40141	.43828	2.2817	.91590	**20**
41	168	862	.2799	578	19
42	195	897	.2781	566	18
43	221	932	.2763	555	17
44	248	.43966	.2745	543	16
45	.40275	.44001	2.2727	.91531	**15**
46	301	036	.2709	519	14
47	328	071	.2691	508	13
48	355	105	.2673	496	12
49	381	140	.2655	484	11
50	.40408	.44175	2.2637	.91472	**10**
51	434	210	.2620	461	9
52	461	244	.2602	449	8
53	488	279	.2584	437	7
54	514	314	.2566	425	6
55	.40541	.44349	2.2549	.91414	**5**
56	567	384	.2531	402	4
57	594	418	.2513	390	3
58	621	453	.2496	378	2
59	647	488	.2478	366	1
60	.40674	.44523	2.2460	.91355	**0**
	N Cos	N Cot	N Tan	N Sin	′

67°—Natural Functions—66°

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24°—Natural Functions—25°

'	N Sin	N Tan	N Cot	N Cos	
0	.40674	.44523	2.2460	.91355	60
1	700	558	.2443	343	59
2	727	593	.2425	331	58
3	753	627	.2408	319	57
4	780	662	.2390	307	56
5	.40806	.44697	2.2373	.91295	55
6	833	732	.2355	283	54
7	860	767	.2338	272	53
8	886	802	.2320	260	52
9	913	837	.2303	248	51
10	.40939	.44872	2.2286	.91236	50
11	966	907	.2268	224	49
12	.40992	942	.2251	212	48
13	.41019	.44977	.2234	200	47
14	045	.45012	.2216	188	46
15	.41072	.45047	2.2199	.91176	45
16	098	082	.2182	164	44
17	125	117	.2165	152	43
18	151	152	.2148	140	42
19	178	187	.2130	128	41
20	.41204	.45222	2.2113	.91116	40
21	231	257	.2096	104	39
22	257	292	.2079	092	38
23	284	327	.2062	080	37
24	310	362	.2045	068	36
25	.41337	.45397	2.2028	.91056	35
26	363	432	.2011	044	34
27	390	467	.1994	032	33
28	416	502	.1977	020	32
29	443	538	.1960	.91008	31
30	.41469	.45573	2.1943	.90996	30
31	496	608	.1926	984	29
32	522	643	.1909	972	28
33	549	678	.1892	960	27
34	575	713	.1876	948	26
35	.41602	.45748	2.1859	.90936	25
36	628	784	.1842	924	24
37	655	819	.1825	911	23
38	681	854	.1808	899	22
39	707	889	.1792	887	21
40	.41734	.45924	2.1775	.90875	20
41	760	960	.1758	863	19
42	787	.45995	.1742	851	18
43	813	.46030	.1725	839	17
44	840	065	.1708	826	16
45	.41866	.46101	2.1692	.90814	15
46	892	136	.1675	802	14
47	919	171	.1659	790	13
48	945	206	.1642	778	12
49	972	242	.1625	766	11
50	.41998	.46277	2.1609	.90753	10
51	.42024	312	.1592	741	9
52	051	348	.1576	729	8
53	077	383	.1560	717	7
54	104	418	.1543	704	6
55	.42130	.46454	2.1527	.90692	5
56	156	489	.1510	680	4
57	183	525	.1494	668	3
58	209	560	.1478	655	2
59	235	595	.1461	643	1
60	.42262	.46631	2.1445	.90631	0
	N Cos	N Cot	N Tan	N Sin	'

'	N Sin	N Tan	N Cot	N Cos	
0	.42262	.46631	2.1445	.90631	60
1	288	666	.1429	618	59
2	315	702	.1413	606	58
3	341	737	.1396	594	57
4	367	772	.1380	582	56
5	.42394	.46808	2.1364	.90569	55
6	420	843	.1348	557	54
7	446	879	.1332	545	53
8	473	914	.1315	532	52
9	499	950	.1299	520	51
10	.42525	.46985	2.1283	.90507	50
11	552	.47021	.1267	495	49
12	578	056	.1251	483	48
13	604	092	.1235	470	47
14	631	128	.1219	458	46
15	.42657	.47163	2.1203	.90446	45
16	683	199	.1187	433	44
17	709	234	.1171	421	43
18	736	270	.1155	408	42
19	762	305	.1139	396	41
20	.42788	.47341	2.1123	.90383	40
21	815	377	.1107	371	39
22	841	412	.1092	358	38
23	867	448	.1076	346	37
24	894	483	.1060	334	36
25	.42920	.47519	2.1044	.90321	35
26	946	555	.1028	309	34
27	972	590	.1013	296	33
28	.42999	626	.0997	284	32
29	.43025	662	.0981	271	31
30	.43051	.47698	2.0965	.90259	30
31	077	733	.0950	246	29
32	104	769	.0934	233	28
33	130	805	.0918	221	27
34	156	840	.0903	208	26
35	.43182	.47876	2.0887	.90196	25
36	209	912	.0872	183	24
37	235	948	.0856	171	23
38	261	.47984	.0840	158	22
39	287	.48019	.0825	146	21
40	.43313	.48055	2.0809	.90133	20
41	340	091	.0794	120	19
42	366	127	.0778	108	18
43	392	163	.0763	095	17
44	418	198	.0748	082	16
45	.43445	.48234	2.0732	.90070	15
46	471	270	.0717	057	14
47	497	306	.0701	045	13
48	523	342	.0686	032	12
49	549	378	.0671	019	11
50	.43575	.48414	2.0655	.90007	10
51	602	450	.0640	.89994	9
52	628	486	.0625	981	8
53	654	521	.0609	968	7
54	680	557	.0594	956	6
55	.43706	.48593	2.0579	.89943	5
56	733	629	.0564	930	4
57	759	665	.0549	918	3
58	785	701	.0533	905	2
59	811	737	.0518	892	1
60	.43837	.48773	2.0503	.89879	0
	N Cos	N Cot	N Tan	N Sin	'

65°—Natural Functions—64°

′	N Sin	N Tan	N Cot	N Cos	
0	.43837	.48773	2.0503	.89879	**60**
1	863	809	.0488	867	59
2	889	845	.0473	854	58
3	916	881	.0458	841	57
4	942	917	.0443	828	56
5	.43968	.48953	2.0428	.89816	**55**
6	.43994	.48989	.0413	803	54
7	.44020	.49026	.0398	790	53
8	046	062	.0383	777	52
9	072	098	.0368	764	51
10	.44098	.49134	2.0353	.89752	**50**
11	124	170	.0338	739	49
12	151	206	.0323	726	48
13	177	242	.0308	713	47
14	203	278	.0293	700	46
15	.44229	.4931$\bar{5}$	2.0278	.89687	**45**
16	25$\bar{5}$	351	.0263	674	44
17	281	387	.0248	662	43
18	307	423	.0233	649	42
19	333	459	.0219	636	41
20	.44359	.49495	2.0204	.89623	**40**
21	385	532	.0189	610	39
22	411	568	.0174	597	38
23	437	604	.0160	584	37
24	464	640	.014$\bar{5}$	571	36
25	.44490	.49677	2.0130	.89558	**35**
26	516	713	.0115	545	34
27	542	749	.0101	532	33
28	568	786	.0086	519	32
29	594	822	.0072	506	31
30	.44620	.49858	2.0057	.89493	**30**
31	646	894	.0042	480	29
32	672	931	.0028	467	28
33	698	.49967	2.0013	454	27
34	724	.50004	1.9999	441	26
35	.447$\bar{5}$0	.50040	1.9984	.89428	**25**
36	776	076	.9970	415	24
37	802	113	.9955	402	23
38	828	149	.9941	389	22
39	854	185	.9926	376	21
40	.44880	.50222	1.9912	.89363	**20**
41	906	258	.9897	350	19
42	932	29$\bar{5}$	.9883	337	18
43	958	331	.9868	324	17
44	.44984	368	.9854	311	16
45	.45010	.50404	1.9840	.89298	**15**
46	036	441	.9825	28$\bar{5}$	14
47	062	477	.9811	272	13
48	088	514	.9797	259	12
49	114	550	.9782	245	11
50	.45140	.50587	1.9768	.89232	**10**
51	166	623	.9754	219	9
52	192	660	.9740	206	8
53	218	696	.9725	193	7
54	243	733	.9711	180	6
55	.45269	.50769	1.9697	.89167	**5**
56	295	806	.9683	153	4
57	321	843	.9669	140	3
58	347	879	.9654	127	2
59	373	916	.9640	114	1
60	.45399	.50953	1.9626	.89101	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.45399	.50953	1.9626	.89101	**60**
1	42$\bar{5}$	.50989	.9612	087	59
2	451	.51026	.9598	074	58
3	477	063	.9584	061	57
4	503	099	.9570	048	56
5	.45529	.51136	1.9556	.8903$\bar{5}$	**55**
6	554	173	.9542	021	54
7	580	209	.9528	.89008	53
8	606	246	.9514	.8899$\bar{5}$	52
9	632	283	.9$\bar{5}$00	981	51
10	45658	.51319	1.9486	.88968	**50**
11	684	356	.9472	95$\bar{5}$	49
12	710	393	.9458	942	48
13	736	403	.9444	928	47
14	762	467	.9430	915	46
15	.45787	.51503	1.9416	.88902	**45**
16	813	540	.9402	888	44
17	839	577	.9388	875	43
18	86$\bar{5}$	614	.937$\bar{5}$	862	42
19	891	651	.9361	848	41
20	.45917	.51688	1.9347	.88835	**40**
21	942	724	.9333	822	39
22	968	761	.9319	808	38
23	.45994	798	.9306	79$\bar{5}$	37
24	.46020	835	.9292	782	36
25	.46046	.51872	1.9278	.88768	**35**
26	072	909	.926$\bar{5}$	75$\bar{5}$	34
27	097	946	.9251	741	33
28	123	.51983	.9237	728	32
29	149	.52020	.9223	71$\bar{5}$	31
30	.4617$\bar{5}$	.52057	1.9210	.88701	**30**
31	201	094	.9196	688	29
32	226	131	.9183	674	28
33	252	168	.9169	661	27
34	278	20$\bar{5}$	.9155	647	26
35	.46304	.52242	1.9142	.88634	**25**
36	330	279	.9128	620	24
37	355	316	.911$\bar{5}$	607	23
38	381	353	.9101	593	22
39	407	390	.9088	580	21
40	.46433	.52427	1.9074	.88566	**20**
41	458	464	.9061	553	19
42	484	501	.9047	539	18
43	510	538	.9034	526	17
44	536	575	.9020	512	16
45	.46561	.52613	1.9007	.88499	**15**
46	587	6$\bar{5}$0	.8993	485	14
47	613	687	.8980	472	13
48	639	724	.8967	458	12
49	664	761	.8953	44$\bar{5}$	11
50	.46690	.52798	1.8940	.88431	**10**
51	716	836	.8927	417	9
52	742	873	.8913	404	8
53	767	910	.8900	390	7
54	793	947	.8887	377	6
55	.46819	.5298$\bar{5}$	1.8873	.88363	**5**
56	844	.53022	.8860	349	4
57	870	059	.8847	336	3
58	896	096	.8834	322	2
59	921	134	.8820	308	1
60	.46947	.53171	1.8807	.8829$\bar{5}$	**0**
	N Cos	N Cot	N Tan	N Sin	

63°—Natural Functions—62°

′	N Sin	N Tan	N Cot	N Cos	
0	.46947	.53171	1.8807	.88295	**60**
1	973	208	.8794	281	59
2	.46999	246	.8781	267	58
3	.47024	283	.8768	254	57
4	050	320	.8755	240	56
5	.47076	.53358	1.8741	.88226	**55**
6	101	395	.8728	213	54
7	127	432	.8715	199	53
8	153	470	.8702	185	52
9	178	507	.8689	172	51
10	.47204	.53545	1.8676	.88158	**50**
11	229	582	.8663	144	49
12	255	620	.8650	130	48
13	281	657	.8637	117	47
14	306	694	.8624	103	46
15	.47332	.53732	1.8611	88089	**45**
16	358	769	.8598	075	44
17	383	807	.8585	062	43
18	409	844	.8572	048	42
19	434	882	.8559	034	41
20	.47460	.53920	1.8546	.88020	**40**
21	486	957	.8533	.88006	39
22	511	.53995	.8520	.87993	38
23	537	.54032	.8507	979	37
24	562	070	.8495	965	36
25	.47588	.54107	1.8482	.87951	**35**
26	614	145	.8469	937	34
27	639	183	.8456	923	33
28	665	220	.8443	909	32
29	690	258	.8430	896	31
30	.47716	.54296	1.8418	.87882	**30**
31	741	333	.8405	868	29
32	767	371	.8392	854	28
33	793	409	.8379	840	27
34	818	446	.8367	826	26
35	.47844	.54484	1.8354	.87812	**25**
36	869	522	.8341	798	24
37	895	560	.8329	784	23
38	920	597	.8316	770	22
39	946	635	.8303	756	21
40	.47971	.54673	1.8291	.87743	**20**
41	.47997	711	.8278	729	19
42	.48022	748	.8265	715	18
43	048	786	.8253	701	17
44	073	824	.8240	687	16
45	.48099	.54862	1.8228	.87673	**15**
46	124	900	.8215	659	14
47	150	938	.8202	645	13
48	175	.54975	.8190	631	12
49	201	.55013	.8177	617	11
50	.48226	.55051	1.8165	.87603	**10**
51	252	089	.8152	589	9
52	277	127	.8140	575	8
53	303	165	.8127	561	7
54	328	203	.8115	546	6
55	.48354	.55241	1.8103	.87532	**5**
56	379	279	.8090	518	4
57	405	317	.8078	504	3
58	430	355	.8065	490	2
59	456	393	.8053	476	1
60	.48481	.55431	1.8040	.87462	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.48481	.55431	1.8040	.87462	**60**
1	506	469	.8028	448	59
2	532	507	.8016	434	58
3	557	545	.8003	420	57
4	583	583	.7991	406	56
5	.48608	.55621	1.7979	.87391	**55**
6	634	659	.7966	377	54
7	659	697	.7954	363	53
8	684	736	.7942	349	52
9	710	774	.7930	335	51
10	.48735	.55812	1.7917	.87321	**50**
11	761	850	.7905	306	49
12	786	888	.7893	292	48
13	811	926	.7881	278	47
14	837	.55964	.7868	264	46
15	.48862	.56003	1.7856	.87250	**45**
16	888	041	.7844	235	44
17	913	079	.7832	221	43
18	938	117	.7820	207	42
19	964	156	.7808	193	41
20	.48989	.56194	1.7796	.87178	**40**
21	.49014	232	.7783	164	39
22	040	270	.7771	150	38
23	065	309	.7759	136	37
24	090	347	.7747	121	36
25	.49116	.56385	1.7735	.87107	**35**
26	141	424	.7723	093	34
27	166	462	.7711	079	33
28	192	501	.7699	064	32
29	217	539	.7687	050	31
30	.49242	.56577	1.7675	.87036	**30**
31	268	616	.7663	021	29
32	293	654	.7651	.87007	28
33	318	693	.7639	.86993	27
34	344	731	.7627	978	26
35	.49369	.56769	1.7615	.86964	**25**
36	394	808	.7603	949	24
37	419	846	.7591	935	23
38	445	885	.7579	921	22
39	470	923	.7567	906	21
40	.49495	.56962	1.7556	.86892	**20**
41	521	.57000	.7544	878	19
42	546	039	.7532	863	18
43	571	078	.7520	849	17
44	596	116	.7508	834	16
45	.49622	.57155	1.7496	.86820	**15**
46	647	193	.7485	805	14
47	672	232	.7473	791	13
48	697	271	.7461	777	12
49	723	309	.7449	762	11
50	.49748	.57348	1.7437	.86748	**10**
51	773	386	.7426	733	9
52	798	425	.7414	719	8
53	824	464	.7402	704	7
54	849	503	.7391	690	6
55	.49874	.57541	1.7379	.86675	**5**
56	899	580	.7367	661	4
57	924	619	.7355	646	3
58	950	657	.7344	632	2
59	.49975	696	.7332	617	1
60	.50000	.57735	1.7321	.86603	**0**
	N Cos	N Cot	N Tan	N Sin	′

61°—Natural Functions—60°

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′	N Sin	N Tan	N Cot	N Cos	
0	.50000	.57735	1.7321	.86603	**60**
1	025	774	.7309	588	59
2	050	813	.7297	573	58
3	076	851	.7286	559	57
4	101	890	.7274	544	56
5	.50126	.57929	1.7262	.86530	**55**
6	151	.57968	.7251	515	54
7	176	.58007	.7239	501	53
8	201	046	.7228	486	52
9	227	085	.7216	471	51
10	.50252	.58124	1.7205	.86457	**50**
11	277	162	.7193	442	49
12	302	201	.7182	427	48
13	327	240	.7170	413	47
14	352	279	.7159	398	46
15	.50377	.58318	1.7147	.86384	**45**
16	403	357	.7136	369	44
17	428	396	.7124	354	43
18	453	435	.7113	340	42
19	478	474	.7102	325	41
20	.50503	.58513	1.7090	.86310	**40**
21	528	552	.7079	295	39
22	553	591	.7067	281	38
23	578	631	.7056	266	37
24	603	670	.7045	251	36
25	.50628	.58709	1.7033	.86237	**35**
26	654	748	.7022	222	34
27	679	787	.7011	207	33
28	704	826	.6999	192	32
29	729	865	.6988	178	31
30	.50754	.58905	1.6977	.86163	**30**
31	779	944	.6965	148	29
32	804	.58983	.6954	133	28
33	829	.59022	.6943	119	27
34	854	061	.6932	104	26
35	.50879	.59101	1.6920	.86089	**25**
36	904	140	.6909	074	24
37	929	179	.6898	059	23
38	954	218	.6887	045	22
39	.50979	258	.6875	030	21
40	.51004	.59297	1.6864	.86015	**20**
41	029	336	.6853	.86000	19
42	054	376	.6842	.85985	18
43	079	415	.6831	970	17
44	104	454	.6820	956	16
45	.51129	.59494	1.6808	.85941	**15**
46	154	533	.6797	926	14
47	179	573	.6786	911	13
48	204	612	.6775	896	12
49	229	651	.6764	881	11
50	.51254	.59691	1.6753	.85866	**10**
51	279	730	.6742	851	9
52	304	770	.6731	836	8
53	329	809	.6720	821	7
54	354	849	.6709	806	6
55	.51379	.59888	1.6698	.85792	**5**
56	404	928	.6687	777	4
57	429	.59967	.6676	762	3
58	454	.60007	.6665	747	2
59	479	046	.6654	732	1
60	.51504	.60086	1.6643	.85717	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.51504	.60086	1.6643	.85717	**60**
1	529	126	.6632	702	59
2	554	165	.6621	687	58
3	579	205	.6610	672	57
4	604	245	.6599	657	56
5	.51628	.60284	1.6588	.85642	**55**
6	653	324	.6577	627	54
7	678	364	.6566	612	53
8	703	403	.6555	597	52
9	728	443	.6545	582	51
10	.51753	.60483	1.6534	.85567	**50**
11	778	522	.6523	551	49
12	803	562	.6512	536	48
13	828	602	.6501	521	47
14	852	642	.6490	506	46
15	.51877	.60681	1.6479	.85491	**45**
16	902	721	.6469	476	44
17	927	761	.6458	461	43
18	952	801	.6447	446	42
19	.51977	841	.6436	431	41
20	.52002	.60881	1.6426	.85416	**40**
21	026	921	.6415	401	39
22	051	.60960	.6404	385	38
23	076	.61000	.6393	370	37
24	101	040	.6383	355	36
25	.52126	.61080	1.6372	.85340	**35**
26	151	120	.6361	325	34
27	175	160	.6351	310	33
28	200	200	.6340	294	32
29	225	240	.6329	279	31
30	.52250	.61280	1.6319	.85264	**30**
31	275	320	.6308	249	29
32	299	360	.6297	234	28
33	324	400	.6287	218	27
34	349	440	.6276	203	26
35	.52374	.61480	1.6265	.85188	**25**
36	399	520	.6255	173	24
37	423	561	.6244	157	23
38	448	601	.6234	142	22
39	473	641	.6223	127	21
40	.52498	.61681	1.6212	.85112	**20**
41	522	721	.6202	096	19
42	547	761	.6191	081	18
43	572	801	.6181	066	17
44	597	842	.6170	051	16
45	.52621	.61882	1.6160	.85035	**15**
46	646	922	.6149	020	14
47	671	.61962	.6139	.85005	13
48	696	.62003	.6128	.84989	12
49	720	043	.6118	974	11
50	.52745	.62083	1.6107	.84959	**10**
51	770	124	.6097	943	9
52	794	164	.6087	928	8
53	819	204	.6076	913	7
54	844	245	.6066	897	6
55	.52869	.62285	1.6055	.84882	**5**
56	893	325	.6045	866	4
57	918	366	.6034	851	3
58	943	406	.6024	836	2
59	967	446	.6014	820	1
60	.52992	.62487	1.6003	.84805	**0**
	N Cos	N Cot	N Tan	N Sin	′

59°—Natural Functions—58°

′	N Sin	N Tan	N Cot	N Cos	
0	.52992	.62487	1.6003	.84805̄	**60**
1	.53017	527	.5993	789	59
2	041	568	.5983	774	58
3	066	608	.5972	759	57
4	091	649	.5962	743	56
5	.53115	.62689	1.5952	.84728	**55**
6	140	730	.5941	712	54
7	164	770	.5931	697	53
8	189	811	.5921	681	52
9	214	852	.5911	666	51
10	.53238	.62892	1.5900	.84650	**50**
11	263	933	.5890	635̄	49
12	288	.62973	.5880	619	48
13	312	.63014	.5869	604	47
14	337	055̄	.5859	588	46
15	.53361	.63095	1.5849	.84573	**45**
16	386	136	.5839	557	44
17	411	177	.5829	542	43
18	435	217	.5818	526	42
19	460	258	.5808	511	41
20	.53484	.63299	1.5798	.84495	**40**
21	509	340	.5788	480	39
22	534	380	.5778	464	38
23	558	421	.5768	448	37
24	583	462	.5757	433	36
25	.53607	.63503	1.5747	.84417	**35**
26	632	544	.5737	402	34
27	656	584	.5727	386	33
28	681	625	.5717	370	32
29	705	666	.5707	355̄	31
30	.53730	.63707	1.5697	.84339	**30**
31	754	748	.5687	324	29
32	779	789	.5677	308	28
33	804	830	.5667	292	27
34	828	871	.5657	277	26
35	.53853	.63912	1.5647	.84261	**25**
36	877	953	.5637	245	24
37	902	.63994	.5627	230	23
38	926	.64035̄	.5617	214	22
39	951	076	.5607	198	21
40	.53975	.64117	1.5597	.84182	**20**
41	.54000	158	.5587	167	19
42	024	199	.5577	151	18
43	049	240	.5567	135	17
44	073	281	.5557	120	16
45	.54097	.64322	1.5547	.84104	**15**
46	122	363	.5537	088	14
47	146	404	.5527	072	13
48	171	446	.5517	057	12
49	195	487	.5507	041	11
50	.54220	.64528	1.5497	.84025	**10**
51	244	569	.5487	.84009	9
52	269	610	.5477	.83994	8
53	293	652	.5468	978	7
54	317	693	.5458	962	6
55	.54342	.64734	1.5448	.83946	**5**
56	366	775	.5438	930	4
57	391	817	.5428	915̄	3
58	415	858	.5418	899	2
59	440	899	.5408	883	1
60	.54464	.64941	1.5399	.83867	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.54464	.64941	1.5399	.83867	**60**
1	488	.64982	.5389	851	59
2	513	.65024	.5379	835̄	58
3	537	065̄	.5369	819	57
4	561	106	.5359	804	56
5	.54586	.65148	1.535̄0	.83788	**55**
6	610	189	.5340	772	54
7	635	231	.5330	756	53
8	659	272	.5320	740	52
9	683	314	.5311	724	51
10	.54708	.65355	1.5301	.83708	**50**
11	732	397	.5291	692	49
12	756	438	.5282	676	48
13	781	480	.5272	660	47
14	805̄	521	.5262	645̄	46
15	.54829	.65563	1.5253	.83629	**45**
16	854	604	.5243	613	44
17	878	646	.5233	597	43
18	902	688	.5224	581	42
19	927	729	.5214	565̄	41
20	.54951	.65771	1.5204	.83549	**40**
21	975	813	.5195̄	533	39
22	.54999	854	.5185	517	38
23	.55024	896	.5175	501	37
24	048	938	.5166	485̄	36
25	.55072	.65980	1.5156	.83469	**35**
26	097	.66021	.5147	453	34
27	121	063	.5137	437	33
28	145	105̄	.5127	421	32
29	169	147	.5118	405̄	31
30	.55194	.66189	1.5108	.83389	**30**
31	218	230	.5099	373	29
32	242	272	.5089	356	28
33	266	314	.5080	340	27
34	291	356	.5070	324	26
35	.55315̄	.66398	1.5061	.83308	**25**
36	339	440	.5051	292	24
37	363	482	.5042	276	23
38	388	524	.5032	260	22
39	412	566	.5023	244	21
40	.55436	.66608	1.5013	.83228	**20**
41	460	65̄0	.5004	212	19
42	484	692	.4994	195	18
43	509	734	.4985̄	179	17
44	533	776	.4975	163	16
45	.55557	.66818	1.4966	.83147	**15**
46	581	860	.4957	131	14
47	605	902	.4947	115̄	13
48	630	944	.4938	098	12
49	654	.66986	.4928	082	11
50	.55678	.67028	1.4919	.83066	**10**
51	702	071	.4910	05̄0	9
52	726	113	.4900	034	8
53	750	155̄	.4891	017	7
54	775̄	197	.4882	.83001	6
55	.55799	.67239	1.4872	.82985	**5**
56	823	282	.4863	969	4
57	847	324	.4854	953	3
58	871	366	.4844	936	2
59	895	409	.4835̄	920	1
60	.55919	.67451	1.4826	.82904	**0**
	N Cos	N Cot	N Tan	N Sin	′

57°—Natural Functions—56°

′	N Sin	N Tan	N Cot	N Cos	
0	.55919	.67451	1.4826	.82904	**60**
1	943	493	.4816	887	59
2	968	536	.4807	871	58
3	.55992	578	.4798	855̄	57
4	.56016	620	.4788	839	56
5	.56040	.67663	1.4779	.82822	**55**
6	064	705	.4770	806	54
7	088	748	.4761	790	53
8	112	790	.4751	773	52
9	136	832	.4742	757	51
10	.56160	.67875̄	1.4733	.82741	**50**
11	184	917	.4724	724	49
12	208	.67960	.4715̄	708	48
13	232	.68002	.4705	692	47
14	256	045	.4696	675	46
15	.56280	.68088	1.4687	.82659	**45**
16	305̄	130	.4678	643	44
17	329	173	.4669	626	43
18	353	215	.4659	610	42
19	377	258	.4650	593	41
20	.56401	.68301	1.4641	82577	**40**
21	425̄	343	.4632	561	39
22	449	386	.4623	544	38
23	473	429	.4614	528	37
24	497	471	.4605̄	511	36
25	.56521	.68514	1.4596	.82495̄	**35**
26	545̄	557	.4586	478	34
27	569	600	.4577	462	33
28	593	642	.4568	446	32
29	617	685	.4559	429	31
30	56641	.68728	1.4550	82413	**30**
31	665̄	771	.4541	396	29
32	689	814	.4532	380	28
33	713	857	.4523	363	27
34	736	900	.4514	347	26
35	.56760	.68942	1.4505̄	.82330	**25**
36	784	.68985	.4496	314	24
37	808	.69028	.4487	297	23
38	832	071	.4478	281	22
39	856	114	.4469	264	21
40	.56880	.69157	1.4460	.82248	**20**
41	904	200	.4451	231	19
42	928	243	.4442	214	18
43	952	286	.4433	198	17
44	.56976	329	.4424	181	16
45	.57000	.69372	1.4415̄	.82165̄	**15**
46	024	416	.4406	148	14
47	047	459	.4397	132	13
48	071	502	.4388	115̄	12
49	095	545̄	.4379	098	11
50	.57119	.69588	1.4370	.82082	**10**
51	143	631	.4361	065	9
52	167	675̄	.4352	048	8
53	191	718	.4344	032	7
54	215̄	761	.4335̄	.82015	6
55	.57238	.69804	1.4326	.81999	**5**
56	262	847	.4317	982	4
57	286	891	.4308	965	3
58	310	934	.4299	949	2
59	334	.69977	.4290	932	1
60	.57358	.70021	1.4281	.81915	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.57358	.70021	1.4281	.81915	**60**
1	381	064	.4273	899	59
2	405	107	.4264	882	58
3	429	151	.4255̄	865	57
4	453	194	.4246	848	56
5	.57477	.70238	1.4237	.81832	**55**
6	501	281	.4229	815̄	54
7	524	325̄	.4220	798	53
8	548	368	.4211	782	52
9	572	412	.4202	765̄	51
10	.57596	70455	1.4193	.81748	**50**
11	619	499	.4185̄	731	49
12	643	542	.4176	714	48
13	667	586	.4167	698	47
14	691	629	.4158	681	46
15	.57715̄	.70673	1.415̄0	.81664	**45**
16	738	717	.4141	647	44
17	762	760	.4132	631	43
18	786	804	.4124	614	42
19	810	848	.4115̄	597	41
20	.57833	.70891	1.4106	.81580	**40**
21	857	935	.4097	563	39
22	881	.70979	.4089	546	38
23	904	.71023	.4080	530	37
24	928	066	.4071	513	36
25	.57952	.71110	1.4063	.81496	**35**
26	976	154	.4054	479	34
27	.57999	198	.4045	462	33
28	.58023	242	.4037	445	32
29	047	285	.4028	428	31
30	58070	.71329	1.4019	.81412	**30**
31	094	373	.4011	395̄	29
32	118	417	.4002	378	28
33	141	461	.3994	361	27
34	165̄	505	.3985	344	26
35	.58189	.71549	1.3976	.81327	**25**
36	212	593	.3968	310	24
37	236	637	.3959	293	23
38	260	681	.3951	276	22
39	283	725	.3942	259	21
40	.58307	.71769	1.3934	.81242	**20**
41	330	813	.3925	225	19
42	354	857	.3916	208	18
43	378	901	.3908	191	17
44	401	946	.3899	174	16
45	.58425̄	.71990	1.3891	.81157	**15**
46	449	.72034	.3882	140	14
47	472	078	.3874	123	13
48	496	122	.3865	106	12
49	519	167	.3857	089	11
50	.58543	.72211	1.3848	.81072	**10**
51	567	255	.3840	055	9
52	590	299	.3831	038	8
53	614	344	.3823	021	7
54	637	388	.3814	.81004	6
55	.58661	.72432	1.3806	.80987	**5**
56	684	477	.3798	970	4
57	708	521	.3789	953	3
58	731	565	.3781	936	2
59	755̄	610	.3772	919	1
60	.58779	.72654	1.3764	.80902	**0**
	N Cos	N Cot	N Tan	N Sin	′

55°—Natural Functions—54°

'	N Sin	N Tan	N Cot	N Cos	
0	.58779	.72654	1.3764	.80902	60
1	802	699	.3755	885	59
2	826	743	.3747	867	58
3	849	788	.3739	850	57
4	873	832	.3730	833	56
5	.58896	.72877	1.3722	.80816	55
6	920	921	.3713	799	54
7	943	.72966	.3705	782	53
8	967	.73010	.3697	765	52
9	.58990	055	.3688	748	51
10	.59014	.73100	1.3680	.80730	50
11	037	144	.3672	713	49
12	061	189	.3663	696	48
13	084	234	.3655	679	47
14	108	278	.3647	662	46
15	.59131	.73323	1.3638	.80644	45
16	154	368	.3630	627	44
17	178	413	.3622	610	43
18	201	457	.3613	593	42
19	225	502	.3605	576	41
20	.59248	.73547	1.3597	.80558	40
21	272	592	.3588	541	39
22	295	637	.3580	524	38
23	318	681	.3572	507	37
24	342	726	.3564	489	36
25	.59365	.73771	1.3555	.80472	35
26	389	816	.3547	455	34
27	412	861	.3539	438	33
28	436	906	.3531	420	32
29	459	951	.3522	403	31
30	.59482	.73996	1.3514	.80386	30
31	506	.74041	.3506	368	29
32	529	086	.3498	351	28
33	552	131	.3490	334	27
34	576	176	.3481	316	26
35	.59599	.74221	1.3473	.80299	25
36	622	267	.3465	282	24
37	646	312	.3457	264	23
38	669	357	.3449	247	22
39	693	402	.3440	230	21
40	.59716	.74447	1.3432	.80212	20
41	739	492	.3424	195	19
42	763	538	.3416	178	18
43	786	583	.3408	160	17
44	809	628	.3400	143	16
45	.59832	.74674	1.3392	.80125	15
46	856	719	.3384	108	14
47	879	764	.3375	091	13
48	902	810	.3367	073	12
49	926	855	.3359	056	11
50	.59949	.74900	1.3351	.80038	10
51	972	946	.3343	021	9
52	.59995	.74991	.3335	.80003	8
53	.60019	.75037	.3327	.79986	7
54	042	082	.3319	968	6
55	.60065	.75128	1.3311	.79951	5
56	089	173	.3303	934	4
57	112	219	.3295	916	3
58	135	264	.3287	899	2
59	158	310	.3278	881	1
60	.60182	.75355	1.3270	.79864	0
	N Cos	N Cot	N Tan	N Sin	'

'	N Sin	N Tan	N Cot	N Cos	
0	.60182	.75355	1.3270	.79864	60
1	205	401	.3262	846	59
2	228	447	.3254	829	58
3	251	492	.3246	811	57
4	274	538	.3238	793	56
5	.60298	.75584	1.3230	.79776	55
6	321	629	.3222	758	54
7	344	675	.3214	741	53
8	367	721	.3206	723	52
9	390	767	.3198	706	51
10	.60414	.75812	1.3190	.79688	50
11	437	858	.3182	671	49
12	460	904	.3175	653	48
13	483	950	.3167	635	47
14	506	.75996	.3159	618	46
15	.60529	.76042	1.3151	.79600	45
16	553	088	.3143	583	44
17	576	134	.3135	565	43
18	599	180	.3127	547	42
19	622	226	.3119	530	41
20	.60645	.76272	1.3111	.79512	40
21	668	318	.3103	494	39
22	691	364	.3095	477	38
23	714	410	.3087	459	37
24	738	456	.3079	441	36
25	.60761	.76502	1.3072	.79424	35
26	784	548	.3064	406	34
27	807	594	.3056	388	33
28	830	640	.3048	371	32
29	853	686	.3040	353	31
30	.60876	.76733	1.3032	.79335	30
31	899	779	.3024	318	29
32	922	825	.3017	300	28
33	945	871	.3009	282	27
34	968	918	.3001	264	26
35	.60991	.76964	1.2993	.79247	25
36	.61015	.77010	.2985	229	24
37	038	057	.2977	211	23
38	061	103	.2970	193	22
39	084	149	.2962	176	21
40	.61107	.77196	1.2954	.79158	20
41	130	242	.2946	140	19
42	153	289	.2938	122	18
43	176	335	.2931	105	17
44	199	382	.2923	087	16
45	.61222	.77428	1.2915	.79069	15
46	245	475	.2907	051	14
47	268	521	.2900	033	13
48	291	568	.2892	.79016	12
49	314	615	.2884	.78998	11
50	.61337	.77661	1.2876	.78980	10
51	360	708	.2869	962	9
52	383	754	.2861	944	8
53	406	801	.2853	926	7
54	429	848	.2846	908	6
55	.61451	.77895	1.2838	.78891	5
56	474	941	.2830	873	4
57	497	.77988	.2822	855	3
58	520	.78035	.2815	837	2
59	543	082	.2807	819	1
60	.61566	.78129	1.2799	.78801	0
	N Cos	N Cot	N Tan	N Sin	'

53°—Natural Functions—52°

′	N Sin	N Tan	N Cot	N Cos	
0	.61566	.78129	1.2799	.78801	**60**
1	589	175	.2792	783	59
2	612	222	.2784	765	58
3	63$\bar{5}$	269	.2776	747	57
4	658	316	.2769	729	56
5	.61681	.78363	1.2761	.78711	**55**
6	704	410	.2753	694	54
7	726	457	.2746	676	53
8	749	504	.2738	658	52
9	772	551	.2731	640	51
10	.61795	.78598	1.2723	.78622	**50**
11	818	645	.2715	604	49
12	841	692	.2708	586	48
13	864	739	.2700	568	47
14	887	786	.2693	5$\bar{5}$0	46
15	.61909	.78834	1.268$\bar{5}$	.78532	**45**
16	932	881	.2677	514	44
17	955	928	.2670	496	43
18	.61978	.78975	.2662	478	42
19	.62001	.79022	.265$\bar{5}$	460	41
20	.62024	.79070	1.2647	.78442	**40**
21	046	117	.2640	424	39
22	069	164	.2632	405	38
23	092	212	.2624	387	37
24	11$\bar{5}$	259	.2617	369	36
25	.62138	.79306	1.2609	.78351	**35**
26	160	354	.2602	333	34
27	183	401	.2594	315	33
28	206	449	.2587	297	32
29	229	496	.2579	279	31
30	.62251	.79544	1.2572	.78261	**30**
31	274	591	.2564	243	29
32	297	639	.2557	22$\bar{5}$	28
33	320	686	.2549	206	27
34	342	734	.2542	188	26
35	.62365	.79781	1.2534	.78170	**25**
36	388	829	.2527	152	24
37	411	877	.2519	134	23
38	433	924	.2512	116	22
39	456	.79972	.2504	098	21
40	.62479	.80020	1.2497	.78079	**20**
41	502	067	.2489	061	19
42	524	115	.2482	043	18
43	547	163	.247$\bar{5}$	02$\bar{5}$	17
44	570	211	.2467	.78007	16
45	.62592	.80258	1.2460	.77988	**15**
46	615	306	.2452	970	14
47	638	354	.244$\bar{5}$	952	13
48	660	402	.2437	934	12
49	683	4$\bar{5}$0	.2430	916	11
50	.62706	.80498	1.2423	.77897	**10**
51	728	546	.2415	879	9
52	751	594	.2408	861	8
53	774	642	.2401	843	7
54	796	690	.2393	824	6
55	.62819	.80738	1.2386	.77806	**5**
56	842	786	.2378	788	4
57	864	834	.2371	769	3
58	887	882	.2364	751	2
59	909	930	.2356	733	1
60	.62932	.80978	1.2349	.7771$\bar{5}$	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.62932	.80978	1.2349	.7771$\bar{5}$	**60**
1	95$\bar{5}$	.81027	.2342	696	59
2	.62977	07$\bar{5}$	.2334	678	58
3	.63000	123	.2327	660	57
4	022	171	.2320	641	56
5	.63045	.81220	1.2312	.77623	**55**
6	068	268	.230$\bar{5}$	60$\bar{5}$	54
7	090	316	.2298	586	53
8	113	364	.2290	568	52
9	135	413	.2283	5$\bar{5}$0	51
10	.63158	.81461	1.2276	.77531	**50**
11	180	510	.2268	513	49
12	203	558	.2261	494	48
13	225	606	.2254	476	47
14	248	65$\bar{5}$	.2247	458	46
15	.63271	.81703	1.2239	.77439	**45**
16	293	752	.2232	421	44
17	316	800	.222$\bar{5}$	402	43
18	338	849	.2218	384	42
19	361	898	.2210	366	41
20	.63383	.81946	1.2203	.77347	**40**
21	406	.8199$\bar{5}$	.2196	329	39
22	428	.82044	.2189	310	38
23	451	092	.2181	292	37
24	473	141	.2174	273	36
25	.63496	.82190	1.2167	.7725$\bar{5}$	**35**
26	518	238	.2160	236	34
27	540	287	.2153	218	33
28	563	336	.2145	199	32
29	585	38$\bar{5}$	.2138	181	31
30	.63608	.82434	1.2131	.77162	**30**
31	630	483	.2124	144	29
32	653	531	.2117	125	28
33	675	580	.2109	107	27
34	698	629	.2102	088	26
35	.63720	.82678	1.2095	.77070	**25**
36	742	727	.2088	051	24
37	76$\bar{5}$	776	.2081	033	23
38	787	825	.2074	.77014	22
39	810	874	.2066	.76996	21
40	.63832	.82923	1.2059	.76977	**20**
41	854	.82972	.2052	959	19
42	877	.83022	.2045	940	18
43	899	071	.2038	921	17
44	922	120	.2031	903	16
45	.63944	.83169	1.2024	.76884	**15**
46	966	218	.2017	866	14
47	.63989	268	.2009	847	13
48	.64011	317	.2002	828	12
49	033	366	.1995	810	11
50	.64056	.83415	1.1988	.76791	**10**
51	078	46$\bar{5}$	.1981	772	9
52	100	514	.1974	754	8
53	123	564	.1967	735	7
54	14$\bar{5}$	613	.1960	717	6
55	.64167	.83662	1.1953	.76698	**5**
56	190	712	.1946	679	4
57	212	761	.1939	661	3
58	234	811	.1932	642	2
59	256	860	.192$\bar{5}$	623	1
60	.64279	.83910	1.1918	.76604	**0**
	N Cos	N Cot	N Tan	N Sin	′

51°—Natural Functions—50°

′	N Sin	N Tan	N Cot	N Cos	
0	.64279	.83910	1.1918	.76604	**60**
1	301	.83960	.1910	586	59
2	323	.84009	.1903	567	58
3	346	059	.1896	548	57
4	368	108	.1889	530	56
5	.64390	.84158	1.1882	.76511	**55**
6	412	208	.1875	492	54
7	435	258	.1868	473	53
8	457	307	.1861	455	52
9	479	357	.1854	436	51
10	.64501	.84407	1.1847	.76417	**50**
11	524	457	.1840	398	49
12	546	507	.1833	380	48
13	568	556	.1826	361	47
14	590	606	.1819	342	46
15	.64612	.84656	1.1812	.76323	**45**
16	635	706	.1806	304	44
17	657	756	.1799	286	43
18	679	806	.1792	267	42
19	701	856	.1785	248	41
20	.64723	.84906	1.1778	.76229	**40**
21	746	.84956	.1771	210	39
22	768	.85006	.1764	192	38
23	790	057	.1757	173	37
24	812	107	.1750	154	36
25	.64834	.85157	1.1743	.76135	**35**
26	856	207	.1736	116	34
27	878	257	.1729	097	33
28	901	308	.1722	078	32
29	923	358	.1715	059	31
30	.64945	.85408	1.1708	.76041	**30**
31	967	458	.1702	022	29
32	.64989	509	.1695	.76003	28
33	.65011	559	.1688	.75984	27
34	033	609	.1681	965	26
35	.65055	.85660	1.1674	.75946	**25**
36	077	710	.1667	927	24
37	100	761	.1660	908	23
38	122	811	.1653	889	22
39	144	862	.1647	870	21
40	.65166	.85912	1.1640	.75851	**20**
41	188	.85963	.1633	832	19
42	210	.86014	.1626	813	18
43	232	064	.1619	794	17
44	254	115	.1612	775	16
45	.65276	.86166	1.1606	.75756	**15**
46	298	216	.1599	738	14
47	320	267	.1592	719	13
48	342	318	.1585	700	12
49	364	368	.1578	680	11
50	.65386	.86419	1.1571	.75661	**10**
51	408	470	.1565	642	9
52	430	521	.1558	623	8
53	452	572	.1551	604	7
54	474	623	.1544	585	6
55	.65496	.86674	1.1538	.75566	**5**
56	518	725	.1531	547	4
57	540	776	.1524	528	3
58	562	827	.1517	509	2
59	584	878	.1510	490	1
60	.65606	.86929	1.1504	.75471	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.65606	.86929	1.1504	.75471	**60**
1	628	.86980	.1497	452	59
2	650	.87031	.1490	433	58
3	672	082	.1483	414	57
4	694	133	.1477	395	56
5	.65716	.87184	1.1470	.75375	**55**
6	738	236	.1463	356	54
7	759	287	.1456	337	53
8	781	338	.1450	318	52
9	803	389	.1443	299	51
10	.65825	.87441	1.1436	.75280	**50**
11	847	492	.1430	261	49
12	869	543	.1423	241	48
13	891	595	.1416	222	47
14	913	646	.1410	203	46
15	.65935	.87698	1.1403	.75184	**45**
16	956	749	.1396	165	44
17	.65978	801	.1389	146	43
18	.66000	852	.1383	126	42
19	022	904	.1376	107	41
20	.66044	.87955	1.1369	.75088	**40**
21	066	.88007	.1363	069	39
22	088	059	.1356	050	38
23	109	110	.1349	030	37
24	131	162	.1343	.75011	36
25	.66153	.88214	1.1336	.74992	**35**
26	175	265	.1329	973	34
27	197	317	.1323	953	33
28	218	369	.1316	934	32
29	240	421	.1310	915	31
30	.66262	.88473	1.1303	.74896	**30**
31	284	524	.1296	876	29
32	306	576	.1290	857	28
33	327	628	.1283	838	27
34	349	680	.1276	818	26
35	.66371	.88732	1.1270	.74799	**25**
36	393	784	.1263	780	24
37	414	836	.1257	760	23
38	436	888	.1250	741	22
39	458	940	.1243	722	21
40	.66480	.88992	1.1237	.74703	**20**
41	501	.89045	.1230	683	19
42	523	097	.1224	664	18
43	545	149	.1217	644	17
44	566	201	.1211	625	16
45	.66588	.89253	1.1204	.74606	**15**
46	610	306	.1197	586	14
47	632	358	.1191	567	13
48	653	410	.1184	548	12
49	675	463	.1178	528	11
50	.66697	.89515	1.1171	.74509	**10**
51	718	567	.1165	489	9
52	740	620	.1158	470	8
53	762	672	.1152	451	7
54	783	725	.1145	431	6
55	.66805	.89777	1.1139	.74412	**5**
56	827	830	.1132	392	4
57	848	883	.1126	373	3
58	870	935	.1119	353	2
59	891	.89988	.1113	334	1
60	.66913	.90040	1.1106	.74314	**0**
	N Cos	N Cot	N Tan	N Sin	′

49°—Natural Functions—48°

′	N Sin	N Tan	N Cot	N Cos	
0	.66913	.90040	1.1106	.74314	**60**
1	935̄	093	.1100	295	59
2	956	146	.1093	276	58
3	978	199	.1087	256	57
4	.66999	251	.1080	237	56
5	.67021	.90304	1.1074	.74217	**55**
6	043	357	.1067	198	54
7	064	410	.1061	178	53
8	086	463	.1054	159	52
9	107	516	.1048	139	51
10	.67129	.90569	1.1041	.74120	**50**
11	151	621	.1035̄	100	49
12	172	674	.1028	080	48
13	194	727	.1022	061	47
14	215	781	.1016	041	46
15	.67237	.90834	1.1009	.74022	**45**
16	258	887	.1003	.74002	44
17	280	940	.0996	.73983	43
18	301	.90993	.0990	963	42
19	323	.91046	.0983	944	41
20	.67344	.91099	1.0977	.73924	**40**
21	366	153	.0971	904	39
22	387	206	.0964	885̄	38
23	409	259	.0958	865	37
24	430	313	.0951	846	36
25	.67452	.91366	1.0945	.73826	**35**
26	473	419	.0939	806	34
27	495̄	473	.0932	787	33
28	516	526	.0926	767	32
29	538	580	.0919	747	31
30	.67559	.91633	1.0913	.73728	**30**
31	580	687	.0907	708	29
32	602	740	.0900	688	28
33	623	794	.0894	669	27
34	645̄	847	.0888	649	26
35	.67666	.91901	1.0881	.73629	**25**
36	688	.91955̄	.0875̄	610	24
37	709	.92008	.0869	590	23
38	730	062	.0862	570	22
39	752	116	.0856	551	21
40	.67773	.92170	1.085̄0	.73531	**20**
41	795̄	224	.0843	511	19
42	816	277	.0837	491	18
43	837	331	.0831	472	17
44	859	385	.0824	452	16
45	.67880	.92439	1.0818	.73432	**15**
46	901	493	.0812	413	14
47	923	547	.0805	393	13
48	944	601	.0799	373	12
49	965	655	.0793	353	11
50	.67987	.92709	1.0786	.73333	**10**
51	.68008	763	.0780	314	9
52	029	817	.0774	294	8
53	051	872	.0768	274	7
54	072	926	.0761	254	6
55	.68093	.92980	1.0755	.73234	**5**
56	115̄	.93034	.0749	215̄	4
57	136	088	.0742	195̄	3
58	157	143	.0736	175	2
59	179	197	.0730	155	1
60	.68200	.93252	1.0724	.73135	**0**
	N Cos	N Cot	N Tan	N Sin	′

′	N Sin	N Tan	N Cot	N Cos	
0	.68200	.93252	1.0724	.73135	**60**
1	221	306	.0717	116	59
2	242	360	.0711	096	58
3	264	415̄	.0705̄	076	57
4	285̄	469	.0699	056	56
5	.68306	.93524	1.0692	.73036	**55**
6	327	578	.0686	.73016	54
7	349	633	.0680	.72996	53
8	370	688	.0674	976	52
9	391	742	.0668	957	51
10	.68412	.93797	1.0661	.72937	**50**
11	434	852	.0655	917	49
12	455̄	906	.0649	897	48
13	476	.93961	.0643	877	47
14	497	.94016	.0637	857	46
15	.68518	.94071	1.0630	.72837	**45**
16	539	125	.0624	817	44
17	561	180	.0618	797	43
18	582	235	.0612	777	42
19	603	290	.0606	757	41
20	.68624	.94345	1.0599	.72737	**40**
21	645	400	.0593	717	39
22	666	455	.0587	697	38
23	688	510	.0581	677	37
24	709	565	.0575̄	657	36
25	.68730	.94620	1.0569	.72637	**35**
26	751	676	.0562	617	34
27	772	731	.0556	597	33
28	793	786	.0550	577	32
29	814	841	.0544	557	31
30	.68835	.94896	1.0538	.72537	**30**
31	857	.94952	.0532	517	29
32	878	.95007	.0526	497	28
33	899	062	.0519	477	27
34	920	118	.0513	457	26
35	.68941	.95173	1.0507	.72437	**25**
36	962	229	.0501	417	24
37	68983	284	.0495̄	397	23
38	69004	340	.0489	377	22
39	025	395	.0483	357	21
40	.69046	.95451	1.0477	.72337	**20**
41	067	506	.0470	317	19
42	088	562	.0464	297	18
43	109	618	.0458	277	17
44	130	673	.0452	257	16
45	.69151	.95729	1.0446	.72236	**15**
46	172	785̄	.0440	216	14
47	193	841	.0434	196	13
48	214	897	.0428	176	12
49	235	.95952	.0422	156	11
50	.69256	.96008	1.0416	.72136	**10**
51	277	064	.0410	116	9
52	298	120	.0404	095	8
53	319	176	.0398	075	7
54	340	232	.0392	055	6
55	.69361	.96288	1.0385	.72035̄	**5**
56	382	344	.0379	.72015̄	4
57	403	400	.0373	.71995̄	3
58	424	457	.0367	974	2
59	445̄	513	.0361	954	1
60	.69466	.96569	1.0355	.71934	**0**
	N Cos	N Cot	N Tan	N Sin	′

44°—Natural Functions

′	N Sin	N Tan	N Cot	N Cos	
0	.69466	.96569	1.0355	.71934	**60**
1	487	625	.0349	914	59
2	508	681	.0343	894	58
3	529	738	.0337	873	57
4	549	794	.0331	853	56
5	.69570	.96850	1.0325	.71833	**55**
6	591	907	.0319	813	54
7	612	.96963	.0313	792	53
8	633	.97020	.0307	772	52
9	654	076	.0301	752	51
10	.69675̄	.97133	1.0295	.71732	**50**
11	696	189	.0289	711	49
12	717	246	.0283	691	48
13	737	302	.0277	671	47
14	758	359	.0271	650	46
15	.69779	.97416	1.0265	.71630	**45**
16	800	472	.0259	610	44
17	821	529	.0253	590	43
18	842	586	.0247	569	42
19	862	643	.0241	549	41
20	.69883	.97700	1.0235	.71529	**40**
21	904	756	.0230	508	39
22	925̄	813	.0224	488	38
23	946	870	.0218	468	37
24	966	927	.0212	447	36
25	.69987	.97984	1.0206	.71427	**35**
26	.70008	.98041	.0200	407	34
27	029	098	.0194	386	33
28	049	155	.0188	366	32
29	070	213	.0182	345	31
30	.70091	.98270	1.0176	.71325	**30**
31	112	327	.0170	305̄	29
32	132	384	.0164	284	28
33	153	441	.0158	264	27
34	174	499	.0152	243	26
35	.70195̄	.98556	1.0147	.71223	**25**
36	215	613	.0141	203	24
37	236	671	.0135̄	182	23
38	257	728	.0129	162	22
39	277	786	.0123	141	21
40	.70298	.98843	1.0117	.71121	**20**
41	319	901	.0111	100	19
42	339	.98958	.0105	080	18
43	360	.99016	.0099	059	17
44	381	073	.0094	039	16
45	.70401	.99131	1.0088	.71019	**15**
46	422	189	.0082	.70998	14
47	443	247	.0076	978	13
48	463	304	.0070	957	12
49	484	362	.0064	937	11
50	.70505̄	.99420	1.0058	.70916	**10**
51	525	478	.0052	896	9
52	546	536	.0047	875	8
53	567	594	.0041	855̄	7
54	587	652	.0035̄	834	6
55	.70608	.99710	1.0029	.70813	**5**
56	628	768	.0023	793	4
57	649	826	.0017	772	3
58	670	884	.0012	752	2
59	690	.99942	.0006	731	1
60	.70711	1.0000	1.0000	.70711	**0**
	N Cos	N Cot	N Tan	N Sin	′

45°—Natural Functions

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Power-factor Table

Phase angle (θ)	cos θ or power factor	sin θ or reactive factor	Phase angle (θ)	cos θ or power factor	sin θ or reactive factor
0°	1.00	0	60°	0.50	0.8660
8° 7′	0.99	0.1411	60° 40′	0.49	0.8717
11° 29′	0.98	0.1990	61° 19′	0.48	0.8773
14° 4′	0.97	0.2431	61° 58′	0.47	0.8827
16° 16′	0.96	0.2800	62° 37′	0.46	0.8879
18° 12′	0.95	0.3122	63° 15′	0.45	0.8929
19° 57′	0.94	0.3412	63° 54′	0.44	0.8980
21° 34′	0.93	0.3676	64° 32′	0.43	0.9028
23° 4′	0.92	0.3919	65° 10′	0.42	0.9075
24° 30′	0.91	0.4146	65° 48′	0.41	0.9121
25° 51′	0.90	0.4359	66° 25′	0.40	0.9165
27° 8′	0.89	0.4560	67° 3′	0.39	0.9218
28° 21′	0.88	0.4750	67° 40′	0.38	0.9250
29° 33′	0.87	0.4931	68° 17′	0.37	0.9290
30° 41′	0.86	0.5103	68° 54′	0.36	0.9330
31° 47′	0.85	0.5268	69° 31′	0.35	0.9367
32° 52′	0.84	0.5426	70° 7′	0.34	0.9404
33° 54′	0.83	0.5578	70° 44′	0.33	0.9440
34° 55′	0.82	0.5724	71° 20′	0.32	0.9474
35° 54′	0.81	0.5864	71° 56′	0.31	0.9507
36° 52′	0.80	0.6000	72° 33′	0.30	0.9539
37° 49′	0.79	0.6131	73° 9′	0.29	0.9570
38° 45′	0.78	0.6259	73° 44′	0.28	0.9600
39° 39′	0.77	0.6380	74° 20′	0.27	0.9629
40° 32′	0.76	0.6499	74° 56′	0.26	0.9656
41° 25′	0.75	0.6614	75° 31′	0.25	0.9682
42° 16′	0.74	0.6726	76° 7′	0.24	0.9708
43° 7′	0.73	0.6834	76° 42′	0.23	0.9732
43° 57′	0.72	0.6940	77° 17′	0.22	0.9755
44° 46′	0.71	0.7042	77° 53′	0.21	0.9777
45° 34′	0.70	0.7142	78° 28′	0.20	0.9798
46° 22′	0.69	0.7238	79° 3′	0.19	0.9818
47° 9′	0.68	0.7332	79° 38′	0.18	0.9837
47° 56′	0.67	0.7424	80° 13′	0.17	0.9854
48° 42′	0.66	0.7513	80° 48′	0.16	0.9871
49° 27′	0.65	0.7599	81° 22′	0.15	0.9887
50° 12′	0.64	0.7684	81° 57′	0.14	0.9902
50° 57′	0.63	0.7766	82° 32′	0.13	0.9915
51° 41′	0.62	0.7846	83° 6′	0.12	0.9928
52° 25′	0.61	0.7924	83° 41′	0.11	0.9939
53° 8′	0.60	0.8000	84° 16′	0.10	0.9950
53° 51′	0.59	0.8074	84° 50′	0.09	0.9959
54° 33′	0.58	0.8146	85° 25′	0.08	0.9968
55° 15′	0.57	0.8216	85° 59′	0.07	0.9975
55° 57′	0.56	0.8285	86° 34′	0.06	0.9982
56° 38′	0.55	0.8352	87° 8′	0.05	0.9987
57° 19′	0.54	0.8417	87° 42′	0.04	0.9992
58° 0′	0.53	0.8480	88° 17′	0.03	0.9995
58° 40′	0.52	0.8542	88° 51′	0.02	0.9998
59° 20′	0.51	0.8602	89° 26′	0.01	0.9999

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(Numbers refer to sections)

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